European Journal of Operational Research 265 (2018) 962–974

Contents lists available at ScienceDirect

European Journal of Operational Research

journal homepage: www.elsevier.com/locate/ejor

Production, Manufacturing and Logistics

On the unimodality of the price-setting newsvendor problem with additive demand under risk considerations

Javier Rubio-Herrero[a,*], Melike Baykal-Gürsoy[b]

[a] Department of Engineering, St. Mary's University, One Camino Santa Maria, San Antonio, TX 78228, United States
[b] Industrial and Systems Engineering Department, CAIT, RUTCOR, Rutgers University, 96 Frelinghuysen Rd, Piscataway, NJ 08854, United States

ARTICLE INFO

Article history:
Received 9 September 2016
Accepted 28 August 2017
Available online 1 September 2017

Keywords:
Inventory
Pricing
Revenue management
Newsvendor
Risk analysis

ABSTRACT

We present a mean–variance analysis of the single-product, single-period, price-setting newsvendor problem with additive, price-dependent demand. The main goal of this paper is to use a mean–variance framework to solve any risk-sensitive instance and find conditions under which the unimodality of the problem is guaranteed. We introduce such conditions via the lost sales rate elasticity, the elasticity of the optimal price, and the elasticity of the expected safety stock surplus to provide managerial insight in terms of the newsvendor's level of service. We also simplify the optimization problem in case that those conditions do not hold. The main contribution of this paper is that, by evaluating the unimodality of the problem for any possible risk attitude, it extends previously published results found for the concavity of the solution in risk-neutral and moderately risk-sensitive cases.

1. Introduction

Optimal pricing and stocking policies are commonplace. It is often that managers have to make decisions in the face of an uncertain outcome and they search for the best information available before making them. This is probably the main reason of the persistence of a problem that has been investigated for over a century: the newsvendor problem. The problem addresses this issue, not only in the classical supply chain framework, but in many other fields where optimal decisions have to be made in regards to a scarce resource that is subject to random demand. This is the case in energy dispatching (Densing, 2013), nurse staffing (Olivares, Terwiesch, & Cassorla, 2008), revenue management in the airline industry (Deshpande & Arikan, 2012), etc. In fact, the newsvendor problem is not only persisting, but it is thriving.

The solution to a newsvendor model can become very complicated very easily. Theoretically simple problems in the sense of their number of variables often present levels of complexity that make it difficult to derive closed-form solutions. While it is not the aim of this paper to detail the numerous developments rendered to this problem, we will give a brief overview of some remarkable work that has been performed in the last few years.

The most basic newsvendor models consider only one controllable variable, that is, they seek the maximization of the profit by only manipulating the stock quantity. The literature of this type of problems, as well as its applications, is vast and readers interested in literature reviews can refer to Khouja (1999). For a long time it was the norm to consider the newsvendor problem as a risk-free problem, meaning that maximization of the expected profit was the only criterion by which the optimization of the stock quantity was performed. Several authors, however, began to introduce new concepts that also took into account the variability of the profit or that looked into minimizing the probability of an extremely averse monetary outcome. This approach is well founded on *risk theory*, a topic for which in turn there is extensive literature available and that gave way to very different risk measures. Examples of such risk measures are the *spectral risk measures* (Acerbi, 2002), a subset of which are the *coherent measures of risk* (Artzner, Delbaen, Eber, & Heath, 1999). Research works that used coherent measures of risk such as *conditional value at risk* (CVar) have been applied in the last years to stock optimization in the newsvendor problem (Ahmed, Çakmak, & Shapiro, 2007; Choi & Ruszczyński, 2008). Other authors have investigated optimality results using risk measures that are not coherent, such as the *value at risk* (VaR) (Özler, Tan, & Karaesmen, 2009) or the much more classic mean–variance analysis (Chen & Federgruen, 2000; Choi, Li, & Yan, 2008; Wu, Li, Wang, & Cheng, 2009), to which we turn our attention to in this paper.

Until the 1950s there used to be a disconnect between inventory managers and economists. For example, the classic re-

* Corresponding author.
E-mail addresses: jrubioherrero@stmarytx.edu (J. Rubio-Herrero), gursoy@soe.rutgers.edu (M. Baykal-Gürsoy).

http://dx.doi.org/10.1016/j.ejor.2017.08.055
0377-2217/

sult of the economic lot sizing problem did not consider a price-dependent demand. In other words, the demand was assumed to be given or, in the best-case scenario, to be a realization of a random variable. It was not until the 1950s that the effect of price on the stochastic demand was introduced (Whitin, 1955). This fact, in turn led the decision makers to use the price as a decision variable to the problem in addition to the stocking quantity. In such models the price of the good in question is no longer a parameter but it needs to be optimized along with the stock quantity and the demand is not only random, but also price-dependent (see e.g., Petruzzi & Dada, 1999). Such dependence takes on the form introduced in Young (1978):

$$D(p, \epsilon) = g(p)\epsilon + y(p),$$

where both $g(\cdot)$ and $y(\cdot)$ model nonincreasing, twice differentiable functions of p. If $g(p) \equiv 1$ and $y(p) \equiv a - bp$ with a, $b > 0$, the demand is said to be in additive form. On the other hand, if $y(p) \equiv 0$ and $g(p) \equiv ap^{-b}$ with a, $b > 0$, the demand is said to be in multiplicative form. Although Petruzzi and Dada (1999) give some conditions for the unimodality (and therefore for the existence of a unique solution) of the price-setting newsvendor problem under additive and multiplicative demand models, some authors have researched in more depth how the uncertainty of the demand affects the optimal solution in risk-neutral settings (Federgruen & Heching, 1999; Mantrala & Raman, 1999; Wang, Jiang, & Shen, 2004; Xu, Chen, & Xu, 2010; Xu, Cai, & Chen, 2011). Kocabıyıkoğlu and Popescu (2011) introduce a risk-neutral unified model in which the unimodality of both the additive and the multiplicative ($b < 1$, i.e. inelastic products that are price-isoelastic) demand cases can be analyzed with a new concept: the *lost sales rate (LSR) elasticity*. Their results are central to those derived in our paper.

As commented before, we turn our attention to the mean–variance analysis, but this time we focus on the single period, single product, price-setting newsvendor problem with price-dependent demand. The mean–variance analysis has its origin in portfolio optimization (Markowitz, 1952) and is still the most widely used risk measure in industry because of its conceptual simplicity. In the last few years we have witnessed the introduction of other risk measures like VaR or CVar. Despite the desirable properties of these new measures, they can be computationally difficult to implement and do not have a clear advantage over mean–variance settings (Grootveld & Hallerbach, 1999).

The mean–variance framework has already been a means for studying this particular problem: Choi and Chiu (2012) use this risk measure to find the optimal price and the optimal stock sequentially in time in the presence of multiplicative demand. This is typical in the fashion industry, where a stocking decision has to be made first, and then, once the actual realization of the demand has occurred, the retailer has to set a price. However, we are mostly interested in joint optimization of both variables, as done in Rubio-Herrero, Baykal-Gürsoy, and Jaśkiewicz (2015), where the authors analyze the conditions for the concavity of the objective function in goods with additive demand. Since we are interested in the unimodality of the mean–variance objective function, our approach is more comprehensive than that is presented in Rubio-Herrero et al. (2015), as unimodality is a more general result than concavity. However, we ultimately pursue to express our results in terms of the LSR elasticity as well. In short, our goals in this paper are the following: extend the framework presented in Kocabıyıkoğlu and Popescu (2011) and Rubio-Herrero et al. (2015) by considering the unimodality of the objective function in the price-setting newsvendor problem with additive demand; find conditions that guarantee unimodality for any risk-averse or risk-seeking setting, while preserving the generality of the results by using very mild assumptions; and give the conditions in terms of three different elasticities, thus providing managerial insight to the results. The latter goal is a change with respect to the use of other metrics that have been historically used, like the hazard rate or the generalized hazard rate, that are much more abstract in nature.

In the following lines, we introduce the problem formulation in Section 2. Then, we analyze separately the risk averse and the risk-seeking newsvendor in Section 3 and Section 4, respectively. Section 5 is dedicated to studying how the profit changes with the risk-attitude. We describe our conclusions in Section 6. All proofs can be found in the appendix.

2. Problem formulation

Consider a retailer that aims at maximizing her expected profit while keeping the variance of the profit under control. This retailer sells a good over a single period. This product may or may not be perishable. In the latter case, she may sell back the excess of stock at a salvage value. Without loss of generality, we assume that the good is perishable and does not have a salvage value. If there exists a salvage value, it can be incorporated by just a change of variables (Choi & Ruszczyński, 2008). In any case, the decision maker decides how much product to buy from the wholesaler at a given cost and sets a price that this good will sell for. Since the demand is uncertain, so is the profit, but she is interested in setting both price and stock quantity in a way that satisfies her sensitivity to risk. This sensitivity is modeled according to the following performance measure:

$$\tilde{P}(p, x) = \underbrace{p\mathbb{E}(min\{D(p, \epsilon), x\}) - cx}_{Expected\ \ profit} - \lambda \underbrace{Var(p \cdot min\{D(p, \epsilon), x\})}_{Variance\ \ of\ \ the\ \ profit}, \tag{1}$$

where the variables p and x are the retailer's price set for the good and the stock quantity, respectively. The replenishment cost cx is given by a constant cost of c monetary units per unit of product. We assume that the unit cost is constant and independent of the quantity replenished. The demand, $D(\cdot, \cdot)$, is random and price-dependent and has the following additive form (Petruzzi & Dada, 1999):

$$D(p, \epsilon) = a - bp + \epsilon,$$

where a, $b > 0$ and ϵ is a continuous random variable with finite variance $Var(\epsilon)$. The term $y(p) = a - bp$ is usually referred to as the *riskless demand*. The random variable ϵ has density function $f(\cdot)$ and cumulative distribution function (cdf), $F(\cdot)$.

In our model, λ is a risk parameter that decreases the value of the performance measure in risk-averse cases ($\lambda > 0$) and increases its value in risk-seeking cases ($\lambda < 0$). Risk-seeking instances are much less explored in the research literature because of the historical dominance of risk-averse perspectives. However, the existence of risk-seeking scenarios is justified by Prospect Theory (Kahneman & Tversky, 1979; Levy, 1992; Tversky & Kahneman, 1992), which claims that people are loss-averse rather than risk-averse and that risk-seeking behaviors can be developed in situations where bidders want to recoup important losses. Typically, human beings are risk-seeking (risk-averse) when their financial performance is under (above) their target (Fishburn, 1977) and this behavior has been shown to be transferable to organizations (Bowman, 1982), meaning that companies that are struggling financially tend to accept larger risks in their decision making.

In this paper we make several assumptions that are helpful in developing the conditions that will follow and do not interfere greatly with our goal of creating a general framework where any risk-sensitive instance can be solved easily or, at least, can be simplified: firstly, ϵ is a random variable with finite variance $Var(\epsilon)$ and compact and convex support $[A, B]$, $A < 0$, $B > 0$; also, $F(\cdot)$ is twice differentiable with continuous second derivative; finally, $\mathbb{E}(\epsilon) = 0$.

These assumptions are not restrictive: if ϵ is defined over an open interval, one can always consider an efficient truncation that captures as much information as possible. On the other hand, the vast majority of random variables used in this type of inventory problems have cdf's that are twice differentiable with continuous second derivative and, if their expectation is not 0, then its value can always be included in a. For this reason the last condition holds WLOG.

The range of prices that the retailer will consider is $[c, p_{max}]$: obviously, one will not retail a product at a lower price than its wholesale price; on the other hand, the upper bound is given by the maximum price at which the worst possible realization of the demand is nonnegative, i.e.,

$$p_{max} = \max_{\{p: y(p)+A \geq 0\}} p = \frac{A+a}{b}.$$

On the other hand, for each price p, the stock quantity x will not be smaller than $y(p)+A$ (the minimum demand attainable at price p) and will not be larger than $y(p)+B$ (the maximum demand attainable at price p).

In order to simplify the derivations we will redefine the objective function in terms of the *safety stock* $z = x - y(p)$, that is, the difference between the replenished quantity and the expected (or riskless) demand at price p (Petruzzi & Dada, 1999; Rubio-Herrero et al., 2015; Thowsen, 1975). Since $x \in [y(p)+A, y(p)+B]$, this new variable is contained in the interval $[A, B]$. After this change of variables and some algebraic calculations, we introduce our new performance measure:

$$P(p,z) = p(\mu(z) + y(p)) - c(z + y(p)) - \lambda p^2 \sigma^2(z), \tag{2}$$

where

$$\mu(z) = \mathbb{E}(min\{\epsilon, z\}) = \int_z^B (z-u) f(u)\, du, \qquad z \in [A, B],$$

$$\sigma^2(z) = Var(min\{\epsilon, z\}) = Var(\epsilon) + \int_z^B (z^2 - u^2) f(u)\, du - \left[\int_z^B (z-u) f(u)\, du\right]^2, \qquad z \in [A, B].$$

These two functions of z and their characteristics will play a key part in the development of the conditions that will follow. The function $\mu(\cdot)$ is an increasing ($\mu'(z) = 1 - F(z)$), concave ($\mu''(z) = -f(z)$) function between A and 0. Moreover, the function $\sigma^2(\cdot)$ is an increasing function ($\sigma^{2'}(z) = 2(z - \mu(z))(1 - F(z))$) between 0 and $Var(\epsilon)$. The proceeding sections and subsections are dedicated to finding the conditions that guarantee a unique solution to the problem

$$\max_{\substack{p \in [c, p_{max}] \\ z \in [A,B]}} P(p,z). \tag{3}$$

More specifically, we will look at the conditions for the quasiconcavity (i.e. unimodality) of P. These conditions will be found by means of sequential optimization (Zabel, 1970) and therefore we will follow these steps: 1. Select a safety stock, z, and find the price that maximizes $P(\cdot, z)$ for that value of z, $p^*(z)$; 2. Substitute this closed-form expression of the optimal price in the objective function in order to come up with a function of only one variable, $P(p^*(z), z) = P^*(z)$; 3. Find the safety stock z^* that maximizes $P^*(\cdot)$.

3. Risk-neutral and risk-averse newsvendor ($\lambda \geq 0$)

In order to analyze the risk-averse newsvendor, we set one extra assumption: $y(c) + 2A \geq 0$. This is a mild assumption that forces the riskless demand at face-value c to be, in the worst-case scenario, at least as much as $-2A$. We know that $c \leq p_{max} = (a+A)/b$, then this assumption requires that $p_{max} \geq c - A/b$. In general, perturbations or errors around the expected demand at a given price should not be excessively large and therefore we do not consider this to be a strong condition. The purpose of this assumption is to bound the optimal price from above, as explained in the proof of Lemma 1, which simplifies greatly the shape of the optimal price function and makes the optimization of P more accessible.

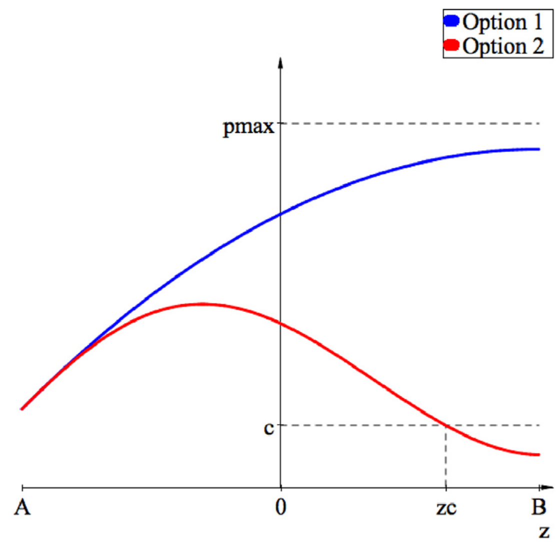

Fig. 1. Typical optimal price functions in risk-averse cases.

3.1. Characteristics of the optimal price

As introduced at the end of Section 2, the first step in our optimization process is to fix a safety stock factor and find the price that maximizes the performance measure. For any $z \in [A, B]$, solving the first-order optimality condition of (2) as a function of p yields a closed form for the optimal price:

$$\frac{\partial P}{\partial p} = 0 \Rightarrow p^*(z) = \frac{\mu(z) + a + cb}{2(\lambda \sigma^2(z) + b)}. \tag{4}$$

This critical point is a maximizer because $\partial^2 P / \partial p^2 = -2(\lambda \sigma^2(z) + b) < 0$ (i.e. $P(\cdot, z)$ is concave with respect to p). Also, clearly, $p^*(z) \leq p^*(z)|_{\lambda=0} = (\mu(z) + a + cb)/(2b)$ and therefore given a safety stock z the optimal price decreases with the level of risk-aversion. It is of great importance to know whether this optimal price is hedged by the interval $[c, p_{max}]$. To that end, the upcoming lemmas and results are intended to shed some light on the shape of this function $p^*(\cdot): [A, B] \to \mathbb{R}$, which is found to be any of the two shown in Fig. 1. In this figure we locate the value $z_c = \min\{\{z : p^*(z) = c\}, B\}$ as either B or as the safety stock that produces an optimal price equal to c in case p^* is unimodal and eventually falls under c.

Lemma 1. *The optimal price $p^*(\cdot)$ is a strictly positive function in $[A, B]$ and $p^*(z) \leq p_{max}$, $\forall z \in [A, B]$.*

Lemma 2. *The optimal price $p^*(\cdot)$ is either a monotonically increasing function in $[A, B)$ or a unimodal function of z.*

Remark 1. Per (A.3) in the appendix, if the optimal price $p^*(\cdot)$ is increasing in a subinterval of $[A, B]$, then it is also concave in that subinterval.

In view of the lemmas above, we can guarantee that the optimal price is not greater than p_{max} but we cannot guarantee that it is not smaller than c. This hindrance is resolved in Rubio-Herrero et al. (2015) by assuming that λ is bounded above by $1/(4Bp_{max})$, which is the minimum value of the right-hand side of (A.2). This assumption is enough to guarantee that $p^*(\cdot)$ is an increasing function in z which, along with the fact that $p^*(A) > c$, is sufficient to conclude that the optimal price is always greater than the replenishment cost. In this case, in order to attain a framework that includes any risk-averse instance, we do not bound the value of the risk parameter and therefore it is possible that the optimal price falls under c. Since we are only concerned about prices in $[c, p_{max}]$, we define the *hedged optimal price function* $\pi^*(\cdot)$ as the following piecewise function:

$$\pi^*(z) = \begin{cases} p^*(z), & \text{if } z \le z_c, \\ c, & \text{if } z > z_c. \end{cases} \tag{5}$$

Clearly this function intends to bound the optimal price within the allowed range of prices in those cases where the risk parameter λ is such that the optimal price eventually falls under the replenishment cost. The performance measure $P(\cdot, z)$ is a concave function with respect to p and this means that $\pi^*(z) = c$ maximizes $P(\cdot, z)$ within the allowed interval $[c, p_{max}]$ whenever $p^*(z) < c$. In general we will use this function to further optimize the performance measure $P(\pi^*(z), z) = P^*(z)$ with respect to z. Nevertheless, there exists a range of nonnegative values for the risk parameter in which $\pi^*(z) = p^*(z),\ \forall z \in [A, B]$. This is shown in the next lemma.

Lemma 3. *The optimal price* $p^*(\cdot)$ *is in* $[c, p_{max}], \forall z \in [A, B]$ *if and only if* $\lambda \in [0, y(c)/(2cVar(\epsilon))]$.

3.2. Optimization of P^*

The next step in our optimization procedure is to redefine the objective function as a function of only the stock factor z. This is achieved by substituting p for the hedged optimal price function. Let us define the following functions of z:

$$P_1^*(z) := P(c, z) = -c^2(\lambda\sigma^2(z) + b) + c(\mu(z) + cb - z),$$

$$P_2^*(z) := P(p^*(z), z) = \frac{1}{2}p^*(z)(\mu(z) + a + cb) - c(z + a).$$

The performance measure at the hedged optimal price $\pi^*(z)$ can be expressed in terms of these two functions above as a continuous piecewise nonlinear function:

$$P^*(z) := P^*(\pi^*(z), z) = \begin{cases} P_2^*(z), & \text{if } z \le z_c, \\ P_1^*(z), & \text{if } z > z_c. \end{cases}$$

The derivative of this function is:

$$P^{*'}(z) = \begin{cases} (1 - F(z))p^*(z)(1 - 2\lambda(z - \mu(z))p^*(z)) - c, & \text{if } z \le z_c. \\ -c^2\lambda\sigma^{2'}(z) - cF(z) \le 0, & \text{if } z > z_c. \end{cases} \tag{6}$$

By taking left and right derivatives at $z = z_c$ (i.e. at the point where $p^*(z) = c$), we can see that $P^*(\cdot)$ is a smooth function (i.e. its first derivative is continuous). Moreover, since $p^*(\cdot)$ is quasi-concave with $0 < p^*(z) \le p_{max}$, and $p^*(A) > c$, it turns out that $\pi^*(\cdot)$ has at most two pieces. Consequently, $P^*(\cdot)$ will have at most two pieces: only $P_2^*(\cdot)$ if $\lambda \in [0, y(c)/(2cVar(\epsilon))]$ (as Lemma 3 dictates for moderately risk-averse situations) or $P_2^*(\cdot)$ and $P_1^*(\cdot)$ (in this order) if $\lambda \in (y(c)/(2cVar(\epsilon)), \infty)$.

Because we will use it in the subsequent sections, we include below the second derivative of the performance measure at the hedged optimal price $\pi^*(z)$ when $\pi^*(z) = p^*(z)$:

$$\begin{aligned} P^{*''}(z)\Big|_{\pi^*(z)=p^*(z)} &= P_2^{*''}(z) = \big(p^{*'}(z)(1 - F(z)) - f(z)p^*(z)\big) \\ &\quad (1 - 2\lambda(z - \mu(z))p^*(z)) \\ &\quad - 2\lambda p^*(z)(1 - F(z))(F(z)p^*(z) \\ &\quad + (z - \mu(z))p^{*'}(z)\big). \end{aligned} \tag{7}$$

Since $P^*(\cdot)$ is a continuous, smooth function and $P_1^*(\cdot)$ is a non-increasing function, it follows that

$$\max_{z \in [A,B]} P^*(z) = \max_{z \in [A, z_c]} P_2^*(z). \tag{8}$$

In other words, the optimal value of the performance measure at the hedged optimal price $\pi^*(z)$ can be found by analyzing only the subinterval in which $\pi^*(z) = p^*(z)$. Therefore, any risk-averse instance can be simplified and reduced to optimizing a nonlinear function instead of a piecewise nonlinear function. On top of this, we will introduce sufficient conditions for the unimodality of the performance measure at the hedged price $\pi^*(\cdot)$. For these conditions, and for some more that will be derived later on, we build our analysis on the *lost sales rate (LSR) elasticity*, the *elasticity of the optimal price*, and the *elasticity of the expected safety stock surplus (ESSS elasticity)*, which we define.

Definition 1. (Kocabıyıkoğlu & Popescu, 2011) The lost sales rate (LSR) elasticity for a given price p and inventory level x is defined as

$$\tilde{\xi}(p, x) = \frac{pG_p(p, x)}{1 - G(p, x)},$$

where $G(p, x) := Pr(D(p, \epsilon) \le x)$ and $G_p(p, x) \equiv \partial G(p, x)/\partial p$.

In the case of additive demand, $Pr(y(p) + \epsilon \le x) = Pr(\epsilon \le x - y(p)) = F(z)$, and we can express this definition as a function of p and z:

$$\tilde{\xi}(p, x) = \frac{pG_p(p, x)}{1 - G(p, x)} = \frac{pbf(z)}{1 - F(z)} =: \xi(p, z). \tag{9}$$

The LSR elasticity represents the percent change in the rate of lost sales with respect to the percent change in price, for a given stock quantity x. This mathematical relationship is explored in depth in Kocabıyıkoğlu and Popescu (2011). Particularizing this expression for the points at which the price is optimal we obtain $\xi(p^*(z), z) := \xi^*(z) = bp^*(z)h(z)$, where $h(z)$ denotes the hazard rate of ϵ, i.e., $h(z) := f(z)/(1 - F(z))$.

Definition 2. The elasticity of the optimal price measures the percentage change in the optimal price when there is a one percent change in the safety stock:

$$\rho^*(z) := \frac{dp^*(z)}{dz}\frac{z}{p^*(z)}. \tag{10}$$

In order to introduce our next elasticity measure, note that the expected safety stock surplus is $E[(z - \epsilon)^+] \equiv z - \mu(z)$.

Definition 3. The elasticity of the expected safety stock surplus (ESSS elasticity) measures the percentage change in the expected excess of safety stock when there is a one percent change in the safety stock:

$$\omega(z) := \frac{d\mathbb{E}\big[(z - \epsilon)^+\big]}{dz}\frac{z}{\mathbb{E}\big[(z - \epsilon)^+\big]}. \tag{11}$$

By definition, the expected safety stock surplus is positive and an increase in the safety stock will inevitably produce an increment in this expectation. Therefore $\omega(\cdot)$ is a nonnegative function.

The addition of these measures to our analysis provides an economic meaning to the conditions that we will present. This makes our analysis much more appealing in managerial environments, as we avoid writing these conditions in terms of more mathematically-oriented terminology and jargon, such as the hazard rate, even though there is a clear connection between this concept and the LSR elasticity. The theorem below presents a set of sufficient conditions for the unimodality of P^*:

Theorem 1. *Let* $\lambda \ge 0,\ z_c = \min\{z : p^*(z) = c, B\}$ *and* $z_\psi = \min\{z : p^{*'}(z) = 0\}$. *If*

$$\xi^*(z) > \frac{1}{2},\ \ \forall z \in [A, z_\psi], \tag{12}$$

Table 1
Example of the conditions for the unimodality of the objective function for risk-averse scenarios.

λ	z_ψ	z_c	Cond. (12)	Cond. (13)	Unimodal?
0	10	10	✓	–	✓
0.01	−2.85	10	✓	✓	✓
0.02	−4.88	10	✓	✓	✓
0.05	−6.71	4.91	✓	✓	✓
0.06	−6.99	3.39	✓	✓	✓

$$-\frac{\rho^*(z)}{\omega(z)} \le 1, \ \forall z \in (z_\psi, z_c], \tag{13}$$

then the performance measure $P^*(\cdot)$ is quasiconcave in $[A, z_c]$ and the price-setting newsvendor problem with additive demand (3) has a unique optimal solution $(z^*, p^*(z^*))$, where z^* solves $P_2^{*'}(z) = 0$ and $p^*(z^*)$ is given by (4).

Simple conditions that guarantee the existence of a unique maximum in $P^*(\cdot)$ have been found in particular cases and have been illustrated in previous publications. For instance, Kocabıyıkoğlu and Popescu (2011) show that in the risk-neutral case, the LSR elasticity has to be at least 1/2 for the objective function to be concave. Similarly, Rubio-Herrero et al. (2015) extend this lower bound for moderate risk-averse cases: when $\lambda < 1/(4Bp_{max})$, the objective function is still unimodal if the LSR elasticity is greater than 1/2. By taking into account the shape of the optimal price function $p^*(\cdot)$ and Theorem 1, we can obtain these bounds as well: in the risk neutral case $(\lambda = 0)$ and in moderately risk-averse cases $(0 < \lambda < 1/(4Bp_{max}))$, $p^*(\cdot)$ is an increasing and concave function, which means that $z_c = z_\psi = B$ and only (12) applies. When the scenario becomes more risk-averse, the optimal price function turns unimodal and there exists now a point $z_\psi < B$ such that $z_\psi = \min\{z : p^{*'}(z) = 0\}$. This point identifies the maximum of $p^*(\cdot)$ and divides the optimal price function into two subintervals: in $[A, z_\psi)$, $p^*(\cdot)$ is increasing and concave; in $[z_\psi, z_c]$, $p^*(\cdot)$ is nonincreasing with a critical point in z_ψ. The particular and predefined shape of this function allows us to propose (12) and (13) and fully characterize the whole spectrum of risk-averse instances.

Example: In order to illustrate how Theorem 1 improves the results obtained by Rubio-Herrero et al. (2015), we will work on the basis of one of the examples provided by the authors. In their paper, they consider the demand function $D(p, \epsilon) = 35 - p + \epsilon$, where $\epsilon \sim U[-10, 10]$. The cost of the commodity is $c = 10$. Under their assumptions, the most risk-averse case under which they can guarantee the concavity of $P^*(\cdot)$ corresponds to a value of the risk parameter $\lambda = 1/(4Bp_{max}) = 1/1400$. With our focus on unimodality, we are able to prove that there is a unique maximum for values of λ beyond 1/1400. In Table 1, we show this for several instances by applying our constant bounds. Fig. 2 displays the application of our constant bounds (12) and (13) to the cases with $\lambda = 0.02$ and $\lambda = 0.06$.

Example: We present now a risk-averse instance that is not unimodal. Consider the demand function $D(p, \epsilon) = 50 - 3p + \epsilon$. Let $c = 5$. The random variable ϵ has a probability density function denoted by $f(z) = 0.5f_1(z) + 0.5f_2(z)$, where $f_1(\cdot)$ and $f_2(\cdot)$ are in turn the pdf's of two normal random variables with means 12 and −12 and variance 0.2. We assume $A = -15$ and $B = 15$ because the density of ϵ beyond those points is negligible. Let $\lambda = 10^{-4}$. It is easy to verify that $z_c = z_\psi = B = 15$ and that the set of conditions (12) and (13) (which can be summarized in this case as $\xi^*(z) > 1/2$) are not met, as shown in Fig. 3. $P^*(\cdot)$ has three critical points.

4. Risk-seeking newsvendor

4.1. Characteristics of the optimal price

When the retailer is risk-seeking, the optimal price $p^*(\cdot)$ presents very different characteristics. From the first-order condition (4), it is easy to see that $p^*(z) < 0$ when $\lambda < -b/\sigma^2(z)$ and exhibits a discontinuity when $\lambda = -b/\sigma^2(z)$. Since $\sigma^2(\cdot)$ is an increasing function, as z increases from A the optimal price reveals three possible, well differentiated pieces, that may appear in the following order: first the optimal price is positive and non-decreasing with respect to z in the region where $\lambda > -b/\sigma^2(z)$; then this price tends to $+\infty$ when $\lambda = -b/\sigma^2(z)$; finally the optimal price surges from $-\infty$ and attains finite negative values when $\lambda < -b/\sigma^2(z)$.

While the first-order optimality condition is the same that was obtained for the risk-averse case, for $z \in [A, B]$ the critical point $p^*(z)$ is not always a maximizer. Indeed, the second partial derivative of P with respect to p

$$\partial^2 P/\partial p^2 = -2(\lambda\sigma^2(z) + b),$$

indicates that the performance measure is concave with respect to p if $\lambda > -b/\sigma^2(z)$, convex with respect to p if $\lambda < -b/\sigma^2(z)$ and linear in p if $\lambda = -b/\sigma^2(z)$. In other words, positive values of $p^*(\cdot)$ correspond to a maximizer of $P(\cdot, z)$, whereas negative values of $p^*(\cdot)$ correspond to a minimizer of the performance measure. In the former case, the concavity of $P(\cdot, z)$ with respect to p when $\lambda > -b/\sigma^2(z)$ implies that the maximizer of $P(\cdot, z)$ in the interval $[c, p_{max}]$ when $p^*(z) > p_{max}$ is obtained at p_{max}. In the latter case, the convexity of $P(\cdot, z)$ with respect to p when $\lambda < -b/\sigma^2(z)$ implies that the maximizer of $P(\cdot, z)$ in the interval $[c, p_{max}]$ is also obtained at p_{max} when $p^*(z) < 0$. This idea is illustrated in Fig. 4, where we chose two different risk scenarios for the same problem and plotted the objective function at $z = 0$. In one scenario, the objective function is concave in p for $z = 0$ and the optimal price is outside of the interval $[c, p_{max}]$ and it is greater than p_{max}. In the other scenario, the objective function is convex in p for $z = 0$ and the optimal price is outside of the interval $[c, p_{max}]$ and it is smaller than 0.

Let $\tilde{z} = \{z : \lambda = -b/\sigma^2(z)\}$ if $\lambda \le -b/Var(\epsilon)$ and $\tilde{z} = B$ if $\lambda > -b/Var(\epsilon)$. Note that when $\lambda \le -b/Var(\epsilon)$ the function p^* is not defined at $z = \tilde{z}$.

Lemma 4. *Let $\lambda < 0$. The optimal price $p^*(\cdot)$ is strictly increasing at all points in $[A, B)$ where it is defined and has a critical point at $z = B$.*

The importance of Lemma 4 is that it gives us a good idea of what $p^*(\cdot)$ looks like. In particular, we know that in many risk-seeking scenarios, the optimal price will go over p_{max}. When that happens, the function will never return to the interval $[c, p_{max}]$. As a matter of fact, only two cases may occur at that point: either the function increases to a point $p^*(B) \ge p_{max}$ or the function presents an asymptote at $z = \tilde{z}$ and $p^*(z) < 0$ in $(\tilde{z}, B]$. Hence, if we let $z_{p_{max}} = \min\{\{z : p^*(z) = p_{max}\}, B\}$ be the safety stock that produces an optimal price equal to p_{max}, or B (whichever is smaller), we can define the hedged optimal price function $\pi(\cdot)$ in the same spirit as in the previous section:

$$\pi^*(z) = \begin{cases} p^*(z), & \text{if } z \le z_{p_{max}}, \\ p_{max}, & \text{if } z > z_{p_{max}}. \end{cases}$$

An illustration of a typical optimal price function and its corresponding hedged optimal price function is presented in Fig. 5. Obtaining $z_{p_{max}}$ will play a central role in characterizing the properties of the objective function $P^*(\cdot)$.

It is also important to understand how $p^*(\cdot)$ changes with the value of the risk parameter. It turns out that, given a safety stock z, $p^*(z)$ always increases as λ decreases: $\partial p^*(z, \lambda)/\partial\lambda =$

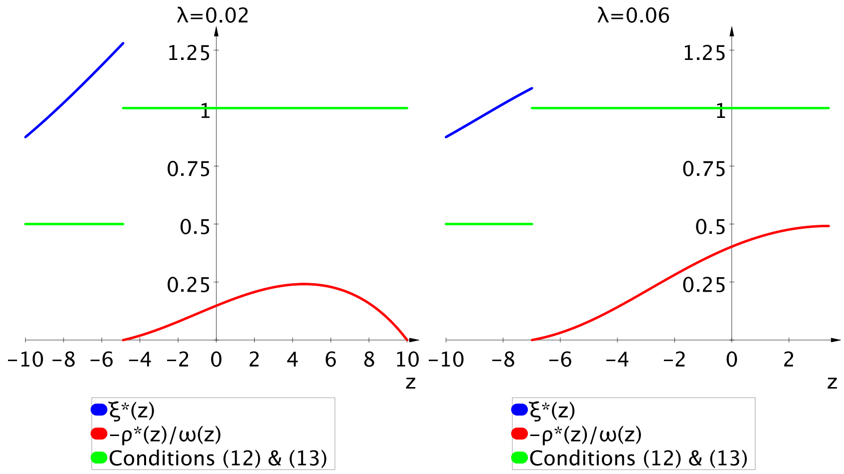

Fig. 2. Conditions (12) and (13) applied to the cases with $\lambda = 0.02$ and $\lambda = 0.06$.

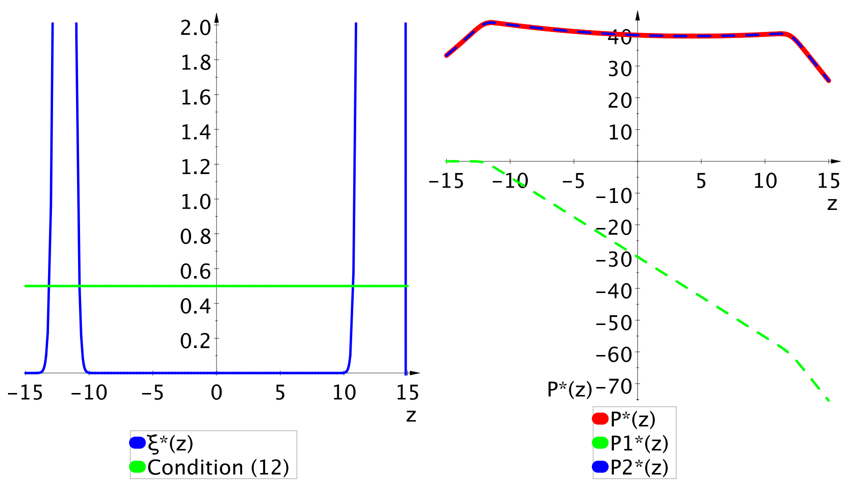

Fig. 3. Multimodal instance of a risk-averse problem.

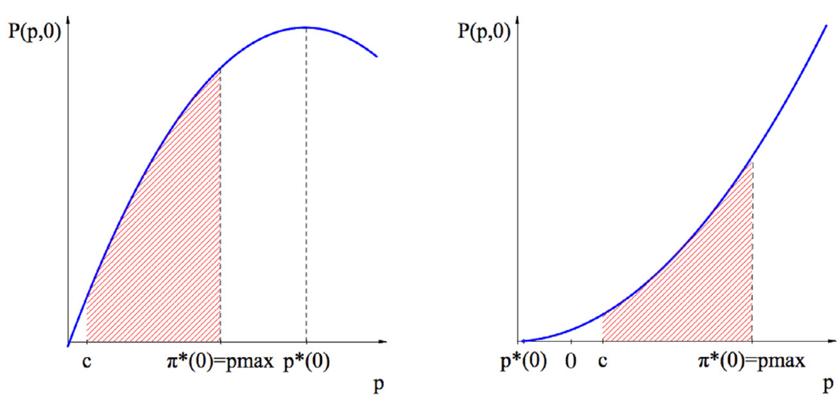

Fig. 4. Obtaining the optimal hedged prices in risk-seeking cases.

$-\sigma^2(z)p^*(z)/\big(\lambda\sigma^2(z)+b\big) \le 0$. It is interesting to see how this impacts the value of $z_{p_{max}}$. In Fig. 6 below we show for a set of values $0 > \lambda_1 > \lambda_2 > \cdots > \lambda_5$ that the value of $z_{p_{max}}$ never increases as we increase the level of risk-seekingness. Indeed $z_{p_{max}}(\lambda_1) = z_{p_{max}}(\lambda_2) = B, z_{p_{max}}(\lambda_3) = z_3, z_{p_{max}}(\lambda_4) = z_4, z_{p_{max}}(\lambda_5) = z_5$.

4.2. Optimization of $P^*(\cdot)$

Let us define the function $P_3^*(z) := P(p_{max}, z) = -p_{max}^2(\lambda\sigma^2(z) + b) + p_{max}(\mu(z) + a + cb) - c(z+a)$. The performance measure at the hedged optimal price $\pi^*(z)$ in the risk-seeking case can be expressed in terms of $P_2^*(\cdot)$ and $P_3^*(\cdot)$ as the following piecewise continuous function:

$$P^*(z) = P^*(\pi^*(z), z) = \begin{cases} P_2^*(z), & \text{if } z \le z_{p_{max}}, \\ P_3^*(z), & \text{if } z > z_{p_{max}}. \end{cases}$$

The derivative of this piecewise, nonlinear function is shown below. Like in the risk-averse case, the left and right derivatives

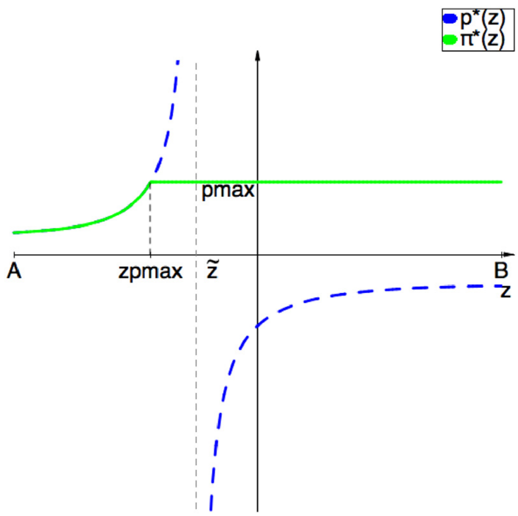

Fig. 5. $p^{*}(\cdot)$ and $\pi^{*}(\cdot)$ in risk-seeking cases.

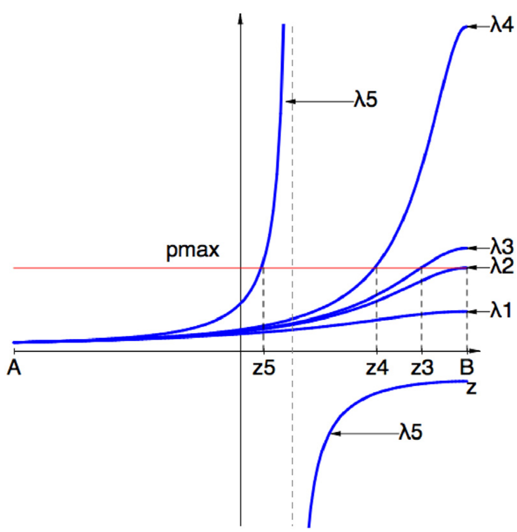

Fig. 6. Optimal price for different levels of risk-seekingness.

of this function at $z = z_{p_{max}}$ are equal and the function is smooth.

$$P^{*'}(z) = \begin{cases} (1 - F(z))p^{*}(z)(1 - 2\lambda(z - \mu(z))p^{*}(z)) - c, & \text{if } z \le z_{p_{max}}, \\ -p_{max}^{2}\lambda\sigma^{2'}(z) + p_{max}(1 - F(z)) - c, & \text{if } z > z_{p_{max}}. \end{cases}$$

Its critical points are attained at $P^{*'}(z) = 0$:

$$P^{*'}(z) = 0 \Rightarrow \begin{cases} (1 - F(z))(1 - 2\lambda(z - \mu(z))p^{*}(z)) = \dfrac{c}{p^{*}(z)}, & \text{if } z \le z_{p_{max}}, \\ F(z) = 1 - \dfrac{c}{p_{max}} - \lambda p_{max}\sigma^{2'}(z), & \text{if } z > z_{p_{max}}. \end{cases} \tag{14}$$

Like in the risk-averse case, where $P_1^{*}(\cdot)$ was always monotonic, the second piece of $P^{*}(\cdot)$, $P_3^{*}(\cdot)$, has a well predefined shape, as shown in the next lemma.

Lemma 5. *The function $P_3^{*}(\cdot)$ is unimodal in $[A, B]$.*

This very specific shape of $P_3^{*}(\cdot)$ makes the analysis much easier, especially if the sign of the slope at the point where this function and $P_2^{*}(\cdot)$ intersect is known. For example, if the slope at the joint is negative, then the mode of $P_3^{*}(\cdot)$ has already occurred when this function becomes part of $P^{*}(\cdot)$, and therefore the maximum of $P^{*}(\cdot)$ will take place in the interval where $P^{*}(z) = P_2^{*}(z)$. Conversely, if the slope at the joint is positive, the maximum of $P_3^{*}(\cdot)$ will occur in the section of $P^{*}(\cdot)$ where $P^{*}(z) = P_3^{*}(z)$. This idea is illustrated in Fig. 7, where, for the sake of generality, $P_2^{*}(\cdot)$ is not always drawn as a unimodal function.

In analyzing the slope at the joint we introduce below two thresholds for λ.

- Let $\lambda_{z_{p_{max}}}$ be the risk parameter that gives way to a scenario where $p^{*}(B) = p_{max}$. In other words, $\lambda_{z_{p_{max}}}$ represents the scenario with the lowest value of λ such that $z_{p_{max}} = B$. More intuitively, the value of $\lambda_{z_{p_{max}}}$ in the case shown in Fig. 6 is λ_2. Analytically, by using (4) we conclude that

$$\lambda_{z_{p_{max}}} = \frac{a + b(c - 2p_{max})}{2p_{max}Var(\epsilon)}.$$

  Any value of λ in the interval $[\lambda_{z_{p_{max}}}, 0]$ will thus produce a hedged optimal price function $\pi^{*}(\cdot)$ equal to the optimal price function $p^{*}(\cdot)$. Any value of λ in the interval $(-\infty, \lambda_{z_{p_{max}}})$ will result in a piecewise hedged optimal price function.
- Let λ_t be the value of the risk parameter that would make $P_2^{*'}(z_{p_{max}}) = P_3^{*'}(z_{p_{max}}) = 0$. By using (14) we conclude that

$$\lambda_t(\lambda) = \begin{cases} -\infty, & \text{if } \lambda \ge \lambda_{z_{p_{max}}}, \\ \dfrac{1 - \dfrac{c}{(1 - F(z_{pmax}(\lambda)))p_{max}}}{2(z_{p_{max}}(\lambda) - \mu(z_{p_{max}}(\lambda)))p_{max}}, & \text{if } \lambda < \lambda_{z_{p_{max}}}. \end{cases}$$

  where we have made it clear that λ_t changes with the level of risk-seekingness λ through the value of $z_{p_{max}}$. The value of λ_t when $\lambda \ge \lambda_{z_{p_{max}}}$ can be obtained from (14) by taking into account that, per definition, $z_{p_{max}}(\lambda) = B$ in these cases.

Clearly $\lambda > \lambda_t(\lambda)$ implies $P_2^{*'}(z_{p_{max}}) = P_3^{*'}(z_{p_{max}}) < 0$. Conversely, $\lambda \le \lambda_t(\lambda)$ implies $P_2^{*'}(z_{p_{max}}) = P_3^{*'}(z_{p_{max}}) \ge 0$. Since λ_t is not a constant threshold value, but rather it changes with λ, in theory the slope at the joint could change its sign several times as we decrease λ. The following lemma shows that this is not the case.

Lemma 6. *There is only one solution to the equation $\lambda = \lambda_t(\lambda)$ in $(-\infty, 0)$.*

This result is important because it means that $P^{*'}(z_{p_{max}}) = 0$ only once in $[A, B]$. Given that $\lambda_t(0) = -\infty$ (because $z_{p_{max}}(0) = B$), this implies that the slope at $z_{p_{max}}$ starts being negative as we decrease λ from a risk-neutral instance. At some risk-seeking instance we attain the equality $\lambda = \lambda_t(\lambda)$ and $P^{*'}(z_{p_{max}}) = 0$. For more risk-seeking settings, we will have a positive slope at the joint between $P_2^{*}(\cdot)$ and $P_3^{*}(\cdot)$. This is shown graphically in Fig. 8.

We can also have some insight about how the critical points of $P_2^{*}(\cdot)$ and $P_3^{*}(\cdot)$ change with λ. To this end we will define, for a given risk parameter λ, the values

$$\zeta_2(\lambda) = \min\left\{z \in [A, B] : P_2^{*'}(z) = 0\right\},$$

$$\zeta_3(\lambda) = \min\left\{z \in [A, B] : P_3^{*'}(z) = 0)\right\}.$$

In other words, $\zeta_2(\lambda)$ and $\zeta_3(\lambda)$ denote the minimum safety stock that produces a critical point in $P_2^{*}(\cdot)$ and $P_3^{*}(\cdot)$, respectively, within the interval $[A, B]$ and for a given risk parameter λ. Since $P_2^{*'}(A) > 0$, $\zeta_2(\lambda)$ always represents a maximum. Since, per Lemma 5, $P_3^{*}(\cdot)$ is unimodal, $\zeta_3(\lambda)$ always represents the unique maximum of this function.

Lemma 7. *Let $\lambda_A < \lambda_B \le 0$. Then $\zeta_2(\lambda_A) > \zeta_2(\lambda_B)$ and $\zeta_3(\lambda_A) > \zeta_3(\lambda_B)$. In other words, the safety stock at which the first maximum in $P_2^{*}(\cdot)$ and the only maximum in $P_3^{*}(\cdot)$ occur over the interval $[A, B]$ increases as λ decreases.*

We introduce now a sufficient condition for $P_2^{*}(\cdot)$ to be unimodal.

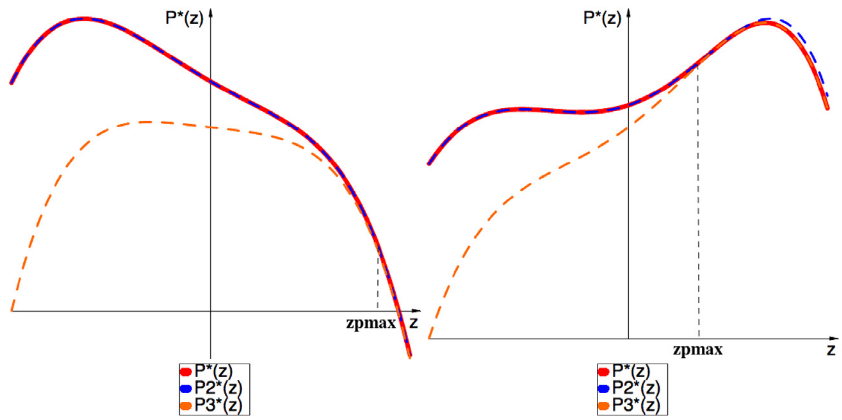

Fig. 7. Importance of the slope at the joint between P_2^* and P_3^*.

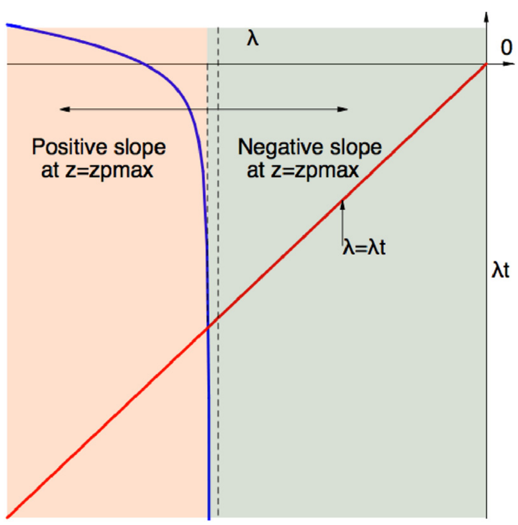

Fig. 8. Explanation of the relationship between λ and $\lambda_t(\lambda)$ to determine the sign of the slope at $z_{p_{max}}$.

Lemma 8. *Let $\lambda \le 0$. If the LSR elasticity $\xi^*(\cdot)$ is bounded below by*

$$\xi^*(z) > \frac{2bc}{A+a+cb} = \frac{c}{p^*(A)}, \quad \forall z \in [A, B], \tag{15}$$

then the performance measure $P_2^*(\cdot)$ is quasiconcave in $[A, B]$.

Corollary 1. *If condition (15) holds for a value of the risk parameter $\hat{\lambda}$ in the interval $(-\infty, 0]$, it also holds for any instance with risk parameter in the interval $(-\infty, \hat{\lambda})$. This follows because $\xi^*(z) = p^*(z)bf(z)/(1-F(z))$ increases with the level of risk-seekingness ($p^*(z)$ increases as λ decreases).*

This lemma and its corollary have two important consequences:

- If the risk-neutral problem is unimodal, then any risk-seeking problem is also unimodal.
- The lower bound provided in (15) is always smaller than 1. When analyzing the risk-neutral case, it can, in some cases, be smaller than the bound of 1/2 provided by Kocabıyıkoğlu and Popescu (2011). Therefore, we can extend their sufficient condition for the concavity of the problem and claim that the quasi-concavity of the risk-neutral problem is guaranteed if

$$\xi^*(z) > \min\left\{\frac{c}{p^*(A)}, \frac{1}{2}\right\}, \quad \forall z \in [A, B].$$

We are now prepared to establish the theorems that tackle the unimodality of the risk-seeking newsvendor problem.

Theorem 2. *Let $\lambda \in [\lambda_t(\lambda), 0)$. Then,*

$$\max_{z\in[A,B]} P^*(z) = \max_{z\in[A,z_{p_{max}}]} P_2^*(z). \tag{16}$$

Moreover, if $P_2^*(\cdot)$ is unimodal in $[A, B]$, then the performance measure $P^*(\cdot)$ is quasiconcave and the price-setting newsvendor problem with additive demand (3) has a unique optimal solution $(\zeta_2(\lambda), p^*(\zeta_2(\lambda)))$, where $\zeta_2(\lambda)$ solves $P_2^{*'}(z) = 0$ and $\pi^*(\zeta_2(\lambda))$ is given by (4).

Theorem 3. *Let $\lambda < 0$ and $\lambda \in (-\infty, \lambda_t(\lambda))$. Then,*

$$\max_{z\in[A,B]} P^*(z) = \max\{P^*(\zeta_3(\lambda)), \max_{z\in[A,z_{p_{max}}]} P_2^*(z)\}. \tag{17}$$

Moreover, if $P_2^*(\cdot)$ is unimodal in $[A, B]$, then the performance measure $P^*(\cdot)$ is quasiconcave and the price-setting newsvendor problem with additive demand (3) has a unique optimal solution $(\zeta_3(\lambda), p^*(\zeta_3(\lambda)))$, where $\zeta_3(\lambda)$ solves $P_3^{*'}(z) = 0$ and $\pi^*(\zeta_3(\lambda))$ is given by (5).

Example: Consider the same demand function as in Section 3, $D(p, \epsilon) = 35 - p + \epsilon$, where $\epsilon \sim U[-10, 10]$. The cost of the commodity is $c = 10$ and $p^*(A) = (A + a + cb)/(2b) = 17.5$. Let us consider two risk-seeking instances: $\lambda_1 = -0.001$ and $\lambda_2 = -0.01$. A simple application of Eq. (15) for the case of $\lambda = 0$ yields the condition $\xi^*(z) > 0.57$, which holds in $[A, B]$ because $\xi^*(A) = 0.875$ and the hazard rate is increasing for a uniform distribution. Per Corollary 1, $P_2^*(\cdot)$ is unimodal in this interval for any risk-seeking instance. For these two scenarios, we obtain that $z_{p_{max}}(-0.001) = B = 10$ and $z_{p_{max}}(-0.01) = 1.28$. These two values yield $\lambda_t(-0.001) = -\infty$ and $\lambda_t(-0.01) = -5.2 \cdot 10^{-4}$. Since $\lambda_t(-0.001) < -0.001$, by virtue of Theorem 2, the only solution to $P_2^{*'}(z) = 0$ provides the triple that solves the problem $(\zeta_2(-0.001) = 2.24, \pi^*(\zeta_2(-0.001)) = 22.11, P^*(\zeta_2(-0.001)) = 108.5$. Since $\lambda_t(-0.01) > -0.01$, by virtue of Theorem 3, the only solution to $P_3^{*'}(z) = 0$ provides the triple that solves the problem $(\zeta_3(-0.01) = 8.48, \pi^*(\zeta_3(-0.01)) = 33.19, P^*(\zeta_3(-0.01)) = 265.58)$. Fig. 9 shows graphically the solution to the instance with $\lambda = -0.01$.

Example: We present now a risk-seeking instance that is not unimodal. Consider the demand function $D(p, \epsilon) = 20 - 1.2p + \epsilon$. Let $c = 6$. The random variable ϵ has a U-quadratic probability density function denoted by $f(z) = 12(u - (B + A)/2)^2/(B - A)^3$ in the interval [6, 6]. Let $\lambda = -0.005$. The sufficient condition for the unimodality of the problem as introduced by (15) is not satisfied in the entire interval $[-6, 6]$, as shown in Fig. 10. In this case $c/p^*(A) = 0.68$. The function $P^*(\cdot)$ has three critical points.

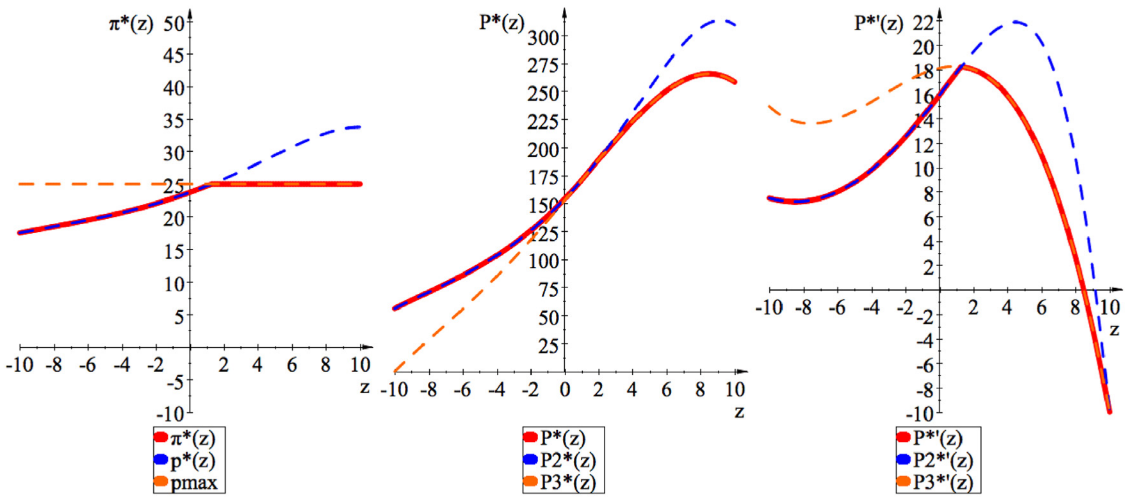

Fig. 9. Example of a risk-seeking newsvendor problem ($D(p,\epsilon)=35-p+\epsilon$, $\epsilon\sim U[-10,10]$, $c=10$, $\lambda=-0.01$).

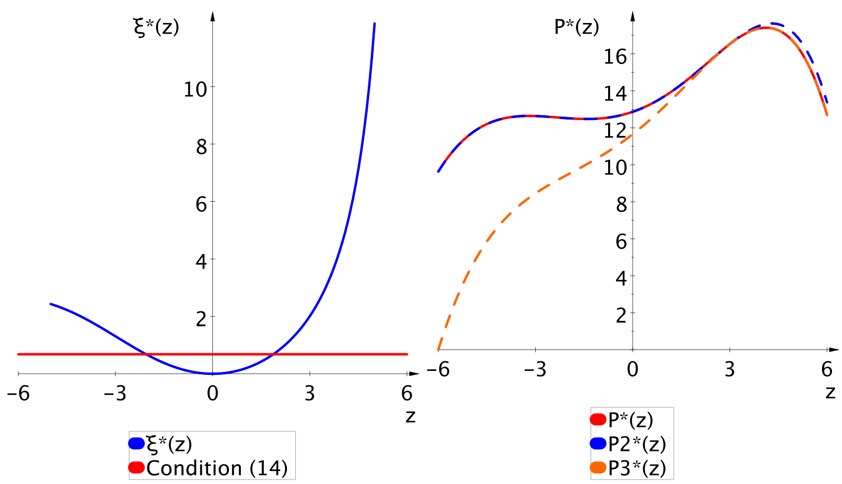

Fig. 10. Multimodal instance of a risk-instance problem.

5. Sensitivity analysis of the expected profit and the variance of the profit

Managerially speaking, the ultimate goal of this analysis is to set the mean and the second central moment (i.e. the variance) of the profit. The selection of an appropriate risk parameter λ is done according to these values and to how acceptable these values are for the decision maker. However, one rarely knows the exact value of the risk parameter that yields the desired expected profit and variance of the profit. If the profit has typically an order of magnitude of $\sim 10^m$, a first approximation can be made by selecting a value of λ that has order of magnitude $\sim 10^{-m}$. Then, the mean–variance tradeoff will present two terms with the same order of magnitude (the variance of the profit is of order $\sim 10^{2m}$). For this reason, we might have to adjust the value of λ in several iterations. This fact makes it important to know beforehand how the expected profit and the variance of the profit will change as a function of the risk parameter. Intuitively, we should expect that increasing the value of λ (i.e. becoming more risk-averse) will reduce the expected profit in exchange for a lower variance of the profit. Likewise, decreasing the value of λ (i.e. becoming more risk-seeking) will reduce the expected profit in exchange for a higher variance of the profit. Our results confirm this behavior.

Lemma 9. *In risk-averse cases, the expected profit and the variance of the profit at the hedged optimal price $\pi^*(z)$ decrease as λ increases. In risk-seeking cases, as λ decreases, the expected profit decreases and the variance of the profit increases.*

Remark 2. Let $\lambda_1>\lambda_2>0$. Then the optimal pair $(z^*_{\lambda_1},\pi^*(z^*_{\lambda_1}))$ produces lower expected profit and a lower variance of the profit than the optimal pair $(z^*_{\lambda_2},\pi^*(z^*_{\lambda_2}))$.

Remark 3. Let $\lambda_1<\lambda_2<0$. Then the optimal pair $(z^*_{\lambda_1},\pi^*(z^*_{\lambda_1}))$ produces lower expected profit and higher variance of the profit than the optimal pair $(z^*_{\lambda_2},\pi^*(z^*_{\lambda_2}))$.

6. Conclusions

The present paper seeks to find a general solution framework and a full characterization for the mean–variance newsvendor problem with price-dependent and additive demand. The performance measure must be seen as a weighted combination of the expected profit and the variance of the profit. The relative importance of the variance of the profit as well as the sign of its contribution to such measure is given by the risk parameter, λ. The decision maker should see this maximization problem as a method to attain optimal stocking and pricing policies in view of his or her risk profile. For each value of λ the maximization problem (3) produces a pair of policies that will in turn yield an expected profit and variance of the profit. These two quantities are ultimately the basis of the decision maker's actions. It is then when she will have to resolve whether these levels of expectation and variance of the profit are acceptable and fine-tune the value of λ accordingly. Fortunately, the model provides intuitive insight on the changes in the first moment and in the second central moment of the profit function: the expected profit and the variance of the profit decrease with the level of risk-aversion, whereas they decrease and increase respectively with the level of risk-seekingness. Therefore tuning the value of λ becomes easier as we know beforehand how it will affect the objective function.

Our results extend the framework of elasticity-related results for additive demand models to fit any risk-sensitive instance, thus complementing those reported by Kocabıyıkoğlu and Popescu (2011) and Rubio-Herrero et al. (2015) for risk-neutral and moderately risk-sensitive cases, respectively. We find that any instance of the risk-sensitive newsvendor problem with mean–variance trade-off can be reformulated as a simplified optimization problem. Moreover, we provide sufficient conditions for the unimodality of risk-sensitive cases in terms of constant bounds of the LSR elasticity, the optimal price elasticity, and ESSS elasticity. While the existence of a unique maximum at a given risk-averse level does not guarantee the existence of a unique maximum for higher risk aversion, we prove that if an instance of the risk-seeking problem is unimodal, then it will also be unimodal for a more risk-seeking case. Finally, we come up with a lower bound of the LSR elasticity for the unimodality of the risk-neutral problem that can, in some cases, be sharper than that proposed by Kocabıyıkoğlu and Popescu (2011) for its concavity.

We want to emphasize that the simplification of this optimization framework for any risk-sensitive instance is possible because the additive demand model yields a closed-form mathematical expression for the optimal price $p^*(\cdot)$. Albeit this price is not necessarily within the range $[c, p_{max}]$ and produces a performance measure $P^*(\cdot)$ that in general is piecewise and nonlinear, its tractability allowed us to characterize $P^*(\cdot)$ fully. This was crucial to develop constant lower bounds for the unimodality of the objective function. It remains a challenge to examine and simplify this mean–variance framework under the light of other demand models whose optimal price functions cannot be obtained in a closed form (e.g. the multiplicative or isoelastic demand model) and to see how the conditions found here change when the newsvendor problem takes place in a multiperiod time horizon. These will be our most imminent research directions.

Appendix A. Proofs of selected theorems and lemmas

Lemma 1

Proof. The first claim is supported by the numerator of (4) being strictly positive since $\mu(z)+a+cb \geq A+a+cb = A+y(c)+2cb > 0$.

To prove that $p^*(z) \leq p_{max}$, we focus first on the risk-neutral case. When $\lambda = 0$, the optimal price p^* is an increasing function in z. Indeed, in this case $p^*(z)|_{\lambda=0} = (\mu(z)+a+cb)/(2b)$ and $p^{*'}(z) = (1-F(z))/(2b) > 0$. Therefore, when $\lambda = 0$ the optimal price has a maximum value $p^*(B) = (a+cb)/(2b)$. This value is smaller than p_{max} because $y(c)+2A \geq 0$. Hence, our only assumption serves the purpose of bounding the optimal price from above.

Given that, per (4), $p^*(z) \leq p^*(z)|_{\lambda=0}$, we conclude that in the risk-averse case $p^*(z) \leq p_{max}$. □

Lemma 2

Proof. The derivative of the optimal price, $p^*(\cdot)$, with respect to the safety stock, z, is

$$p^{*'}(z) = \frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}(1-4\lambda(z-\mu(z))p^*(z)). \tag{A.1}$$

It is not guaranteed that this is positive. As a matter of fact, we have that

$$p^{*'}(z) > 0 \iff \lambda < \frac{1}{4(z-\mu(z))p^*(z)}. \tag{A.2}$$

It is easy to see that $p^{*'}(A) = 1/(2b)$ and $p^{*'}(B) = 0$. Also $p^{*'}(z) = 0$ in (A, B) if and only if there are solutions to the equation $p^*(z) = 1/(4\lambda(z-\mu(z)))$. On the other hand, the second derivative of the optimal price $p^*(\cdot)$ for all $z \in (A, B)$ is

$$p^{*''}(z) = \left(-f(z) - \frac{\lambda\sigma^{2'}(z)(1-F(z))}{\lambda\sigma^2(z)+b}\right)\frac{1-4\lambda(z-\mu(z))p^*(z)}{2(\lambda\sigma^2(z)+b)}$$
$$-4\lambda\frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}(F(z)p^*(z)+(z-\mu(z))p^{*'}(z)), \tag{A.3}$$

which, when evaluated at the points where $p^{*'}(z) = 0$ is

$$p^{*''}(z)|_{p^{*'}(z)=0} = -4\lambda\frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}F(z)p^*(z) < 0. \tag{A.4}$$

Therefore, any critical point that exists in (A, B) is a maximum. Since $p^{*'}(A) > 0$ and $p^{*'}(B) = 0$, the equation $p^*(z) = 1/(4\lambda(z-\mu(z)))$ has **at most** one solution in (A, B) and one of the following outcomes occur: if such a solution does not exist, the function $p^*(\cdot)$ is increasing in $[A, B)$ with a maximum at $z = B$; if such a solution exists at a point z_ψ, the function $p^*(\cdot)$ increases in $[A, z_\psi)$, has a maximum at $z = z_\psi$, decreases in $[z_\psi, B)$, and has an inflection point at $z = B$. It is consequently quasiconcave (unimodal). □

Lemma 3

Proof. By Lemma 1, when $\lambda \geq 0$, $p^*(z) \leq p_{max}, \forall z \in [A, B]$. It remains to validate the conditions for the optimal price to be greater than the replenishment cost.

If we impose in (4) that the optimal price is at least as large as the replenishment cost, it follows that $p^*(z) \geq c$ when $\lambda \leq (\mu(z)+y(c))/(2c\sigma^2(z))$. Therefore this holds for all $z \in [A, B]$ as long as $\lambda \leq \min_{z\in[A,B]}(\mu(z)+y(c))/(2c\sigma^2(z))$.

Let $t(z) = (\mu(z)+y(c))/(2c\sigma^2(z))$. We will prove that this function is decreasing. Its first derivative is $t'(z) = \left((1-F(z))\sigma^2(z) - \sigma^{2'}(z)(\mu(z)+y(c))\right)/\left(2c\left(\sigma^2(z)\right)^2\right)$. While the denominator is always nonnegative, we can also prove that the numerator is nonpositive. Using the equality $\sigma^{2'}(z) = 2(1-F(z))(z-\mu(z))$, note that the numerator is nonpositive if $\sigma^2(z) \leq 2(z-\mu(z))(\mu(z)+y(c))$.

Both sides of this equation are nonnegative in $[A, B]$ and equal to 0 at $z = A$. Moreover,

$$[2(z-\mu(z))(\mu(z)+y(c))]' = 2F(z)(\mu(z)+y(c))$$
$$+\sigma^{2'}(z) \geq \sigma^{2'}(z).$$

Therefore, it is clear that $\sigma^2(z) - 2(z-\mu(z))(\mu(z)+y(c)) \leq 0$ and $t(\cdot)$ is a decreasing function of z. This implies that $p^*(z) \geq c, \forall z \in [A, B]$ if and only if

$$\lambda \leq \min_{z\in[A,B]} t(z) = t(B) = \frac{y(c)}{2cVar(\epsilon)}. \quad \square$$

Theorem 1

Proof. We will analyze $P_2^*(\cdot)$ in $[A,\ z_c]$. P_2^* is a continuous function with $P_2^{*'}(A) = p^*(A) - c > 0$ and $P_2^{*'}(z_c) < 0$. The last inequality follows because in case $P^*(\cdot)$ is a piecewise, nonlinear function, then it is also smooth (i.e. $P_2^*(z_c) = P_1^*(z_c)$) and $P_1^*(\cdot)$ is a decreasing function in $[z_c,\ B]$. Therefore, there must be at least one point in $[A,\ z_c]$ where $P_2^{*'}(z) = 0$. This point is unique and confers quasiconcavity to P_2^* if $P_2^{*''}(z)|_{P_2^{*'}(z)=0} < 0$. Per (6), at the critical points $1 - 2\lambda(z - \mu(z))p^*(z) = c/(p^*(z)(1 - F(z)))$ holds and we can write (7) in terms of the hazard rate $h(z)$ as

$$\frac{cp^{*'}(z)}{p^*(z)} - h(z)c - 2(1 - F(z))\lambda p^*(z)$$
$$\Big(F(z)p^*(z) + (z - \mu(z))p^{*'}(z)\Big) < 0.$$

By using the expression of the LSR elasticity at the optimal price $p^*(z)$ in additive models, $\xi^*(z) = bp^*(z)h(z)$, and observing that $F(z)p^*(z) + (z - \mu(z))p^{*'}(z) = [(z - \mu(z))p^*(z)]'$ we can rewrite the formula above as

$$\xi^*(z) > \left(p^{*'}(z) - \frac{2(1 - F(z))\lambda p^*(z)^2}{c}\right.$$
$$\left.\times\Big(F(z)p^*(z) + (z - \mu(z))p^{*'}(z)\Big)\right). \tag{A.5}$$

Assume that $p^*(\cdot)$ is a unimodal function and consider the two subintervals $[A,\ z_\psi]$ and $(z_\psi,\ z_c]$. We will apply condition (A.5) to both subintervals. In $[A,\ z_\psi]$ the optimal price is nondecreasing and concave with only one critical point at $z = z_\psi$. It follows that the second term inside the parenthesis is nonpositive and the unimodality is guaranteed as long as $\xi^*(z) > bp^{*'}(z) \le bp^{*'}(A) = 1/2, \forall z \in [A, z_\psi]$. In $(z_\psi,\ z_c]$ the optimal price is nonincreasing with only one critical point at $z = z_c$ if $z_c = B$ (otherwise the function is strictly decreasing in $(z_\psi,\ z_c]$). Therefore the first term inside the parenthesis is nonpositive, but the second is only nonpositive if $F(z)p^*(z) + (z - \mu(z))p^{*'}(z) \ge 0$. Per the definition of the ESSS elasticity, note that $\mathbb{E}\Big[(z - \epsilon)^+\Big] = (z - \mu(z))$ and $\Big[\mathbb{E}\Big[(z - \epsilon)^+\Big]\Big]' = F(z)$. Therefore this condition can be written as $\Big[\mathbb{E}\Big[(z - \epsilon)^+\Big]\Big]'p^*(z) + \mathbb{E}\Big[(z - \epsilon)^+\Big]p^{*'}(z) \ge 0$. Applying the definitions of $\omega(\cdot)$ and $\rho^*(\cdot)$ as in (10) and (11) we can guarantee that the second term inside the parenthesis is nonpositive if

$$-\frac{\rho^*(z)}{\omega(z)} \le 1.$$

If this condition applies, then (A.5) is reduced to $\xi^*(z) > 0$ which always holds. □

Lemma 4

Proof. We analyze this function in two subintervals, $[A, \tilde{z})$ and $(\tilde{z}, B]$. Consider Eq. (A.1). Clearly, there is always a critical point at $z = B$. Also, per this equation, when the function $p^*(\cdot)$ is strictly increasing for all z: $p^*(z) > 0$. When $p^*(z) < 0$ (i.e. $\tilde{z} < z \le B$) the function $p^*(\cdot)$ tends to $-\infty$ as we approach $\tilde{z}$ from the right and therefore it is concave and increasing at its right limit towards B, reaching to a value at $z = B$ of $p^*(B) = (a + cb)/(2(\lambda Var(\epsilon) + b))$. Assume it is decreasing in some region in $(\tilde{z}, B)$. Then the function must present a local maximum in such interval. However, this is not possible since, per (A.4), if there is a critical point in $(\tilde{z}, B)$ the function is convex at such point and should be a minimum. Therefore the optimal price $p^*(\cdot)$ has only one critical point at $z = B$, which is also an inflection point. We conclude that $p^*(\cdot)$ is strictly increasing in $(\tilde{z}, B)$. □

Lemma 5

Proof. The function $P_3^*(\cdot)$ has at least one critical point because $P_3^{*'}(A) > 0$ and $P_3^{*'}(B) < 0$. The first-order optimality condition solves the equation

$$\lambda = \frac{p_{max}(1 - F(z)) - c}{p_{max}^2 \sigma^{2'}(z)} = \frac{1 - \dfrac{c}{p_{max}(1 - F(z))}}{2p_{max}^2(z - \mu(z))}.$$

The function on the right-hand side of this equation is decreasing because its denominator increases in z and its numerator decreases in z. Therefore, it attains the constant value λ only once. Given the signs of $P_3^{*'}(A)$ and $P_3^{*'}(B)$, this unique critical point is a maximum. □

Lemma 6

Proof. This result follows easily because $\lim_{\lambda \to \lambda_{z_{p_{max}}}^-} \lambda_t(\lambda) = -\infty$ (remember that $z_{p_{max}}(\lambda) = B$ if $\lambda \in [\lambda_{z_{p_{max}}}, 0)$) and because $\lambda_t(\lambda)$ increases as λ decreases. The latter is easy to see because $z_{p_{max}}$ decreases when λ decreases, which makes the numerator of $\lambda_t(\lambda)$ increase and the denominator decrease as λ decreases. □

Lemma 7

Proof. For clarity of exposition, we will let $\lambda_B = 0$ and $\lambda_A < \lambda_B$, although this result is straightforward to show for any relation $\lambda_A < \lambda_B \le 0$.

Let $\hat{z} = \min\{z : F(z) = 1 - c/p^*(z)|_{\lambda=0}\}$ be the first maximum of the risk-neutral problem. To see that this point is indeed a maximum, consider the risk-neutral problem: in this case $z_{p_{max}} = B$. There is at least a critical point because $P_2^{*'}(A) > 0$, and $P_2^{*'}(B) < 0$. Given the sign of $P_2^{*'}(A)$ this first critical point, $\hat{z}$, will be a maximum.

Compare the first-order optimality conditions of the risk-neutral problem and the risk-seeking problems of the functions $P_2^*(\cdot)$ and $P_3^*(\cdot)$ under the light of the optimal price. Taking into account that $p^*(z) > 0$ whenever $P_2^*(\cdot)$ applies, in any risk-seeking instance and for any safety stock we have that $p^*(z)|_{\lambda=\lambda_A} \ge p^*(z)|_{\lambda=\lambda_B}$. Therefore

$$1 - \frac{c}{p^*(z)|_{\lambda=\lambda_A}\Big(1 - 2\lambda_A(z - \mu(z))p^*(z)|_{\lambda=\lambda_A}\Big)} \ge 1 - \frac{c}{p^*(z)|_{\lambda=\lambda_B}},$$

and

$$1 - \frac{c}{p_{max}} - \lambda_A p_{max}\sigma^{2'}(z) \ge 1 - \frac{c}{p_{max}}.$$

Consequently both first-order conditions will have their first solution at a safety stock higher than $\hat{z}$. This is illustrated in Fig. A.11, where a risk-neutral condition and a risk-seeking condition are shown. □

Lemma 8

Proof. The function $P_2^*(\cdot)$ has at least one solution in $[A,\ B]$ because $P_2^{*'}(A) > 0$ and $P_2^{*'}(B) < 0$. Using (14), consider the equation $P_2^{*'}(z) = 0$. This can be written as

$$\lambda = \frac{1 - \dfrac{c}{p^*(z)(1 - F(z))}}{2(z - \mu(z))p^*(z)}.$$

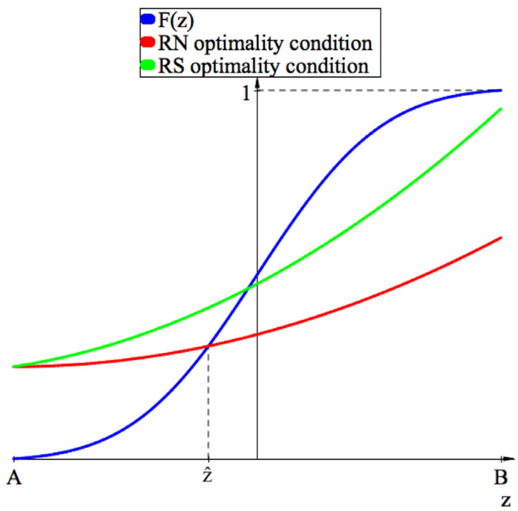

Fig. A.11. Illustration of Lemma 7.

Compare both sides of this equation. The number of times that the function of the right-hand side crosses the constant λ is the number of critical points of $P_2^*(\cdot)$. Since $\lim_{z\to A}(1-c/(p^*(z)(1-F(z))))/(2(z-\mu(z))p^*(z))=\infty$, if this function is always decreasing, it will cross λ exactly once. Taking into account that the denominator $2(z-\mu(z))p^*(z)$ is nondecreasing, it is enough that the numerator is decreasing:

$$\left[1-\frac{c}{p^*(z)(1-F(z))}\right]' = \frac{p^{*'}(z)(1-F(z))-f(z)p^*(z)}{p^*(z)^2(1-F(z))^2},$$

which follows if $p^{*'}(z)(1-F(z))-f(z)p^*(z)<0$ or, in terms of the LSR elasticity, if $\xi^*(z)>bp^{*'}(z)$.

An upper bound for $p^{*'}(\cdot)$ at the critical points is

$$\begin{aligned} p^{*'}(z)\Big|_{P^{*'}(z)=0} &= -\frac{1}{2\left(\lambda\sigma^2(z)+b\right)}\left(1-\frac{2c}{p}-F(z)\right) \\ &\le \frac{c}{p^*(z)\left(\lambda\sigma^2(z)+b\right)} = \frac{2c}{\mu(z)+a+cb} \\ &\le \frac{2c}{A+a+cb}, \end{aligned}$$

whence the condition $\xi^*(z)>c/p^*(A)$ can be derived. □

Theorem 2

Proof. For $\lambda>\lambda_t(\lambda)$, $P^{*'}(z_{p_{max}})<0$. Because of Lemma 5, the function $P_3^*(\cdot)$ is decreasing in $[z_{p_{max}},B]$. Therefore, $\max_{z\in[A,B]}P^*(z)=\max_{z\in[A,z_{p_{max}}]}P_2^*(z)$. If P_2^* is unimodal in $[A,B]$, then its only maximum occurs in the interval $[A,z_{p_{max}}]$ and can be easily attained by solving $P_2^{*'}(z)=0$.

For the case $\lambda=\lambda_t(\lambda)$, $z_{p_{max}}$ is the critical point of $P_3^*(\cdot)$ and a critical point of $P_2^*(\cdot)$. As a result, Eq. (16) still holds, as well as the rest of the theorem. □

Theorem 3

Proof. For $\lambda<\lambda_t(\lambda)$, $P^{*'}(z_{p_{max}})>0$. Because of Lemma 5, the function $P_3^*(\cdot)$ has its only maximum in $[z_{p_{max}},B]$. In general, the function $P_2^*(\cdot)$ may have several critical points in $[A,z_{p_{max}}]$. Therefore, $\max_{z\in[A,B]}P^*(z)=\max\{P^*(\zeta_3(\lambda)),\max_{z\in[A,z_{p_{max}}]}P_2^*(z)\}$.

If $P_2^*(\cdot)$ is unimodal in $[A,B]$, then its only maximum occurs in the interval $[z_{p_{max}},B]$, where $P^*(z)=P_3^*(z)$. Hence, the maximum of $P^*(\cdot)$ is attained at the only point that solves $P_3^{*'}(z)=0$, which is $\zeta_3(\lambda)$. □

Lemma 9

Proof. Let $\Pi^*(z,\lambda)$ be the profit at the hedged price $\pi^*(z,\lambda)$. Recall that the profit is a random variable. From (2) we can redefine our performance measure at the hedged price $\pi^*(z,\lambda)$:

$$P(z,\lambda)=\underbrace{\pi^*(z,\lambda)(\mu(z)+y(\pi^*(z,\lambda)))-c(z+y(\pi^*(z,\lambda)))}_{\mathbb{E}(\Pi^*(z,\lambda))}-\lambda\underbrace{\pi^*(z,\lambda)^2\sigma^2(z)}_{Var(\Pi^*(z,\lambda))}.$$

Consider first the risk-averse case. When $\lambda>0$:

$$\mathbb{E}(\Pi^*(z,\lambda))=\begin{cases} c(\mu(z)-z) & \text{if } z>z_c,\\ p^*(z,\lambda)(\mu(z)+a-bp^*(z,\lambda)) & \\ \quad -c(z+a-bp^*(z,\lambda)) & \text{if } z\le z_c.\end{cases}$$

$$Var(\Pi^*(z,\lambda))=\begin{cases} c^2\sigma^2(z) & \text{if } z>z_c,\\ p^*(z,\lambda)^2\sigma^2(z) & \text{if } z\le z_c.\end{cases}$$

For any given stock factor z the derivative of these two functions are:

$$\frac{\partial}{\partial\lambda}\mathbb{E}(\Pi^*(z,\lambda))=\begin{cases} 0 & \text{if } z>z_c,\\ \dfrac{\partial p^*(z,\lambda)}{\partial\lambda}(\mu(z)+a+b(c-2p^*(z,\lambda))) & \text{if } z\le z_c.\end{cases}$$

$$\frac{\partial}{\partial\lambda}Var(\Pi^*(z,\lambda))=\begin{cases} 0 & \text{if } z>z_c,\\ -\dfrac{2\left(\sigma^2(z)\right)^2}{\lambda\sigma^2(z)+b}p^*(z,\lambda)^2 & \text{if } z\le z_c.\end{cases}$$

Given that $\partial p^*(z,\lambda)/\partial\lambda=-\sigma^2(z)p^*(z,\lambda)/(\lambda\sigma^2(z)+b)\le 0$ and that, per (4), $\mu(z)+a+b(c-2p^*(z,\lambda))>0$ in risk-averse cases, we conclude that the expected profit for a given stock factor at the hedged optimal price does not increase with λ. Also, $\partial Var(\Pi^*(z,\lambda))/\partial\lambda\le 0$ (i.e. as λ increases, the variance of the profit does not increase).

Now consider the risk-seeking case. When $\lambda<0$:

$$\mathbb{E}(\Pi^*(z,\lambda))=\begin{cases} p^*(z,\lambda)(\mu(z)+a-bp^*(z,\lambda) & \\ \quad -c(z+a-bp^*(z,\lambda)) & \text{if } z\le z_{p_{max}},\\ p_{max}(\mu(z)+a-bp_{max}) & \\ \quad -c(z+a-bp_{max}) & \text{if } z>z_{p_{max}}.\end{cases}$$

$$Var(\Pi^*(z,\lambda))=\begin{cases} p^*(z,\lambda)^2\sigma^2(z) & \text{if } z\le z_{p_{max}},\\ p_{max}^2\sigma^2(z) & \text{if } z>z_{p_{max}}.\end{cases}$$

For any given stock factor z the derivative of these two functions are:

$$\frac{\partial}{\partial\lambda}\mathbb{E}(\Pi^*(z,\lambda))=\begin{cases} \dfrac{\partial p^*(z,\lambda)}{\partial\lambda}(\mu(z)+a+b(c-2p^*(z,\lambda))) & \text{if } z\le z_{p_{max}},\\ 0 & \text{if } z>z_{p_{max}}.\end{cases}$$

$$\frac{\partial}{\partial\lambda}Var(\Pi^*(z,\lambda))=\begin{cases} -\dfrac{2\left(\sigma^2(z)\right)^2}{\lambda\sigma^2(z)+b}p^*(z,\lambda)^2 & \text{if } z\le z_{p_{max}},\\ 0 & \text{if } z>z_{p_{max}}.\end{cases}$$

For $z\le z_{p_{max}}$, $\lambda\sigma^2(z)+b\ge 0$ and $p^*(z,\lambda)>c$. Therefore $\partial p^*(z,\lambda)/\partial\lambda=-\sigma^2(z)p^*(z,\lambda)/(\lambda\sigma^2(z)+b)\le 0$. Per (4), $\mu(z)+a+b(c-2p^*(z,\lambda))<0$ in risk-seeking cases, and we conclude that the expected profit for a given stock factor at the hedged optimal price does not decrease with λ (i.e. as λ decreases, the expected profit does not increase). Also, $\partial Var(\Pi^*(z,\lambda))/\partial\lambda\le 0$ (i.e. as λ decreases, the variance of the profit increases). □

References

Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance, 26*(7), 1505–1518.

Ahmed, S., Çakmak, U., & Shapiro, A. (2007). Coherent risk measures in inventory problems. *European Journal of Operational Research, 182*(1), 226–238.

Artzner, P., Delbaen, F., Eber, J.-M., & Heath, D. (1999). Coherent measures of risk. *Mathematical Finance, 9*(3), 203–228.

Bowman, E. H. (1982). Risk seeking by troubled firms. *Sloan Management Review, 23*(4), 33.

Chen, F. & Federgruen, A. (2000). *Mean-variance analysis of basic inventory models*. Technical manuscript, Columbia University.

Choi, S., & Ruszczyński, A. (2008). A risk-averse newsvendor with law invariant coherent measures of risk. *Operations Research Letters, 36*(1), 77–82.

Choi, T.-M., & Chiu, C.-H. (2012). Mean-downside-risk and mean-variance newsvendor models: Implications for sustainable fashion retailing. *International Journal of Production Economics, 135*(2), 552–560.

Choi, T.-M., Li, D., & Yan, H. (2008). Mean–variance analysis for the newsvendor problem. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans, 38*(5), 1169–1180.

Densing, M. (2013). Dispatch planning using newsvendor dual problems and occupation times: Application to hydropower. *European Journal of Operational Research, 228*(2), 321–330.

Deshpande, V., & Arikan, M. (2012). The impact of airline flight schedules on flight delays. *Manufacturing & Service Operations Management, 14*(3), 423–440.

Federgruen, A., & Heching, A. (1999). Combined pricing and inventory control under uncertainty. *Operations Research, 47*(3), 454–475.

Fishburn, P. C. (1977). Mean-risk analysis with risk associated with below-target returns. *The American Economic Review, 67*(2), 116–126.

Grootveld, H., & Hallerbach, W. (1999). Variance vs. downside risk: Is there really that much difference? *European Journal of Operational Research, 114*(2), 304–319.

Kahneman, D., & Tversky, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica: Journal of the Econometric Society, 47*(2), 263–291.

Khouja, M. (1999). The single-period (news-vendor) problem: Literature review and suggestions for future research. *Omega, 27*(5), 537–553.

Kocabıyıkoğlu, A., & Popescu, I. (2011). An elasticity approach to the newsvendor with price-sensitive demand. *Operations Research, 59*(2), 301–312.

Levy, J. S. (1992). An introduction to prospect theory. *Political Psychology, 13*(2), 171–186.

Mantrala, M. K., & Raman, K. (1999). Demand uncertainty and supplier's returns policies for a multi-store style-good retailer. *European Journal of Operational Research, 115*(2), 270–284.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance, 7*(1), 77–91.

Olivares, M., Terwiesch, C., & Cassorla, L. (2008). Structural estimation of the newsvendor model: An application to reserving operating room time. *Management Science, 54*(1), 41–55.

Özler, A., Tan, B., & Karaesmen, F. (2009). Multi-product newsvendor problem with value-at-risk considerations. *International Journal of Production Economics, 117*(2), 244–255.

Petruzzi, N. C., & Dada, M. (1999). Pricing and the newsvendor problem: A review with extensions. *Operations Research, 47*(2), 183–194.

Rubio-Herrero, J., Baykal-Gürsoy, M., & Jaśkiewicz, A. (2015). A price-setting newsvendor problem under mean-variance criteria. *European Journal of Operational Research, 247*(2), 575–587.

Thowsen, G. T. (1975). A dynamic, nonstationary inventory problem for a price/quantity setting firm. *Naval Research Logistics Quarterly, 22*(3), 461–476.

Tversky, A., & Kahneman, D. (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty, 5*(4), 297–323.

Wang, Y., Jiang, L., & Shen, Z.-J. (2004). Channel performance under consignment contract with revenue sharing. *Management Science, 50*(1), 34–47.

Whitin, T. M. (1955). Inventory control and price theory. *Management Science, 2*(1), 61–68.

Wu, J., Li, J., Wang, S., & Cheng, T. (2009). Mean–variance analysis of the newsvendor model with stockout cost. *Omega, 37*(3), 724–730.

Xu, M., Chen, Y. F., & Xu, X. (2010). The effect of demand uncertainty in a price-setting newsvendor model. *European Journal of Operational Research, 207*(2), 946–957.

Xu, X., Cai, X., & Chen, Y. (2011). Unimodality of price-setting newsvendor's objective function with multiplicative demand and its applications. *International Journal of Production Economics, 133*(2), 653–661.

Young, L. (1978). Price, inventory and the structure of uncertain demand. *New Zealand Operations Research, 6*(2), 157–177.

Zabel, E. (1970). Monopoly and uncertainty. *The Review of Economic Studies, 37*(2), 205–219.