Stochastic Decomposition in M/M/∞ queues

with Markov Modulated Service Rates

Melike Baykal-Gursoy[1] and Weihua Xiao[2]

Department of Industrial and Systems Engineering, Rutgers University,

96 Frelinghuysen Road, Piscataway, NJ 08854-8018

[1]*gursoy@rci.rutgers.edu, (732)445-5465, fax (732)445-5465* [2]*xiao@rci.rutgers.edu*

Abstract

Motivated by the need to study transportation systems in which incidents cause traffic to slow down, we consider an M/M/∞ queueing system subject to random interruptions of exponentially distributed durations. System breakdowns, where none of the servers work, as well as partial failures, where all servers work with lower efficiency, are investigated. In both cases, it is shown that the number of customers present in the system in equilibrium is the sum of two independent random variables. One of these is the number of customers present in an ordinary M/M/∞ queue without interruptions.

Keywords: queues; Markov modulated; stochastic decomposition; Kummer functions

1. Introduction

The study of queueing systems with service interruptions has received a significant amount of attention of the researchers in the field. One type of service interruption has already been considered in the context of vacation queues where interruptions only happen as soon as the queue becomes empty. These vacation models are shown to exhibit a stochastic decomposition property. The stationary number of customers in the system can be interpreted as the sum of the state of the corresponding system with no vacations and another nonnegative discrete random variable. A corresponding decomposition result occurs for the waiting time distribution as well. Stochastic decomposition of M/G/1 vacation models has been studied by Cooper [4], Levy and Yechiali [14], Yadin and Naor [20], Fuhrmann [7], Fuhrmann and Cooper [8], Shanthikumar [17], Harris and Marchal [10], and Altiok [1], among many others. A survey of single server queues including GI/G/1 queues with vacations is given in Doshi [5]. Zhang and Tian [21] studied an M/M/C queue with synchronous server vacations, and obtained the stationary distributions of queue length and waiting time in this system using matrix geometric methods. Chao and Zhao [3] considered group server vacations (all servers take vacation and resume service at the same time) and independent vacations (each server takes its own vacation independently of the others) in G/M/C queues, and a computational algorithm was developed to obtain numerical solutions.

Another type of interruptions assumes that service interruptions may happen at any time, and the literature on queues with this type of interruptions is relatively scarce. White and Christe [19] studied a single-server queue with preemptive resume discipline, and related such queues to queues with random service interruptions. Gaver [9], Keilson [12] and Avi-Itzhak and Naor [2] also studied the single-server queue with random interruptions. Gaver [9] obtained the generating functions for the stationary waiting time and the number in the system in an M/G/1 queue. Mitrany and Avi-Itzhak [15] analyzed a multi-server queue where each server may be down independently of the others for an exponential amount of time. They obtained an explicit form of the moment generating function of the queue size for a two-server system, and gave a computational procedure for more than two servers. In the above models, servers fail independently of each other and failures are complete failures such that a failed server becomes completely nonfunctional.

In this paper, we consider an $M/M/\infty$ queueing system subject to batch partial failures, where efficiencies of all servers deteriorate simultaneously at the arrival of an interruption and resume to their normal state when the interruption is cleared. We also consider complete system breakdowns where none of servers work. This problem could also be considered in the context of matrix-geometric queues of Neuts [16], where the servers adhere to Markov-modulated service rate. Keilson and Servi [13] studied a matrix $M/M/\infty$ system in which both the arrival and service processes are Markov-modulated. Ours is a special case of [13], where only the service process is Markov-modulated. We give the complete representation of the stationary distribution of the number of customers in the system.

The motivation for studying this system comes from the field of transportation. Consider a section of a road subject to incidents. The space occupied by an individual vehicle on the road segment represents one queueing "server", which starts its service as soon as a vehicle joins the link and carries the "service" (the act of traveling) until the end of the link is reached. A two-mile roadway section contains hundreds or thousands of such servers, thus an $M/M/\infty$ queueing model is a reasonable approximation. Jain and Smith [11] modeled the traffic flow as an M/G/C/C queue with state dependent service rates. They obtained the steady-state probability of number of vehicles on the road segment. If an incident occurs on the road segment, all the vehicles on the road have to lower their speed until that incident is cleared. The transportation planners would like to estimate the impact of incidents on the traffic flow on a specific road segment in the long run.

The rest of this paper is organized as follows. In section 2, we formulate the problem and describe its queueing model. In section 3, we present the decomposition result and give several important facts about the solution. Details of the proofs are given in Section 4. Finally, in section 5, potential applications of this model and future research are discussed.

2. Mathematical Model

We consider a service system with an infinite number of servers subject to random interruptions of exponentially distributed durations. During interruptions, all servers work at lower efficiency compared to their normal functioning state. The service rate of each server is μ in the absence of interruption, decreases to $\mu' \geq 0$, at the arrival of an interruption, and recovers back to μ at the clearance of the interruption. We assume that interruptions arrive according to a Poisson process with rate f, and the repair time is exponentially distributed with rate r. The customer arrivals are in accordance with a homogeneous Poisson process with intensity λ. The interruption and customer arrival processes and the service and repair times are all assumed to be mutually independent.

The stochastic process $\{X(t), U(t)\}$ describes the state of the system at time t, where $X(t)$ is the number of customers in the system at t, and $U(t)$ is the status of the system. If at time t, the system is experiencing an interruption, then $U(t)$ is equal to F (failure); otherwise, $U(t)$ is N (normal). Keep in mind that the failures considered in this paper are partial failures in the sense that all servers continue to work under deteriorated service rate. The system is said to be in state (i, F) if there are i customers in the system which is damaged by an interruption, while the system is said to be in state (i, N) if there are i customers in the system which is functioning as normal. Accordingly, we denote the steady-state probability of the system being in state (i, F) by $P_{i,F}$ and the steady-state probability of the system being in state (i, N) by $P_{i,N}$.

The steady-state balance equations are given below

$$\begin{aligned}(\lambda + i\mu' + r)P_{i,F} &= (i+1)\mu' P_{i+1,F} + \lambda P_{i-1,F} + fP_{i,N}, \\ (\lambda + i\mu + f)P_{i,N} &= (i+1)\mu P_{i+1,N} + \lambda P_{i-1,N} + rP_{i,F}.\end{aligned} \qquad (i = 1, 2, \ldots) \tag{2.1}$$

The boundary equations are

$$\begin{aligned}(\lambda + r)P_{0,F} &= \mu' P_{1,F} + fP_{0,N}, \\ (\lambda + f)P_{0,N} &= \mu P_{1,N} + rP_{0,F}.\end{aligned} \tag{2.2}$$

Let $G_N(z) = \sum_{i=0}^{\infty} z^i P_{i,N}$ and $G_F(z) = \sum_{i=0}^{\infty} z^i P_{i,F}$, for $|z| \le 1$. Then the generating function of the steady-state number of customers in the system is given by $G(z) = G_F(z) + G_N(z)$.

Multiplying both sides of (2.1) and (2.2) by z^i and summing over all i yield the differential equations

$$\begin{aligned}G_N'(z) &= \frac{1}{(z-1)\mu}[(\lambda z - \lambda - f)G_N(z) + rG_F(z)], \\ G_F'(z) &= \frac{1}{(z-1)\mu'}[(\lambda z - \lambda - r)G_F(z) + fG_N(z)].\end{aligned} \tag{2.3}$$

In the following sections we solve these equations for $G_N(z)$ and $G_F(z)$, and use these to obtain the stationary distribution of the M/M/∞ system.

3. Stochastic Decomposition

We present our decomposition result next. Here X_φ is a Poisson random variable with mean $\varphi = \lambda / \mu$. $\mathscr{B}(a,b,c)$ refers to a truncated beta distribution with parameters, *a, b,* and *c*, its density function is given as

$$\beta(a, b-a, c) = \frac{(\gamma / c)^{a-1}(1 - \gamma / c)^{b-a-1}}{cB(a, b-a)}, \tag{3.1}$$

where $B(a,b-a)=\dfrac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}$ and $\Gamma(a)=\int_0^\infty s^{a-1}e^{-s}ds$ denotes the Euler *gamma function*. Then, the probability mass function of Poisson random variable, Y, randomized by truncated beta, $\mathscr{B}(a,b,c)$, is given by

$$P(Y=k)=\int_{\gamma=0}^{c} e^{-\gamma}\frac{\gamma^k}{k!}\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\frac{(\gamma/c)^{a-1}(1-\gamma/c)^{b-a-1}}{c}d\gamma\ . \tag{3.2}$$

Also, $NB(\phi,\delta)$ refers to a generalized negative binomial distribution with parameters ϕ and δ with generating function, $G(z)=\left(\dfrac{\delta}{1-z(1-\delta)}\right)^{\phi}$. Note that here ϕ could be any real number not necessarily an integer.

Theorem 1: The number of customers in the system, X, in equilibrium has the form

$$X=X_\varphi+Y \tag{3.3}$$

where X_φ and Y are independent, and

$$P\{Y=n\}=pP\{Y_1=n\}+(1-p)P\{Y_2=n\}. \tag{3.4}$$

i) For the case when $\mu'>0$, $p=(r+f\mu'/\mu)/(r+f)$, and Y_1 and Y_2 are conditionally Poisson distributed with random means that have truncated beta distributions $\mathscr{B}(a,b,-2\rho^*)$ and $\mathscr{B}(a+1,b+1,-2\rho^*)$, respectively, where

$$a=\frac{f}{\mu},b=\left(\frac{f}{\mu}+\frac{r}{\mu'}\right),\rho^*=\frac{1}{2}\left(\frac{\lambda}{\mu}-\frac{\lambda}{\mu'}\right). \tag{3.5}$$

The probability mass function of Y_1 and Y_2 are given by (3.2) in terms of their associated parameters.

ii) For the case when $\mu' = 0$, $p = r/(r+f)$, Y_1 is $NB(f/\mu, r/(\lambda + r))$, and Y_2 is $NB((f/\mu)+1, r/(\lambda + r))$.

Proof: The proof is deferred to Section 4.

The following corollary gives the expected value and the variance of the number in the system in equilibrium.

Corollary 1:

i) For the case when $\mu' > 0$, the expected number of customers in equilibrium is given as

$$E(X) = \frac{\lambda}{\mu} + \frac{\lambda f(\mu - \mu')}{\mu^2 (r+f)} \left(1 + \frac{(f+\mu)(\mu - \mu')}{(r\mu + f\mu' + \mu\mu')} \right), \tag{3.6}$$

its variance is derived as

$$\begin{aligned} Var(X) &= \frac{\lambda}{\mu} + \frac{\lambda f(\mu - \mu')}{\mu^2 (r+f)} \left(1 + \frac{(f+\mu)(\mu - \mu')}{(r\mu + f\mu' + \mu\mu')} \right) \\ &+ \frac{\lambda^2}{\mu^2} \left[\begin{array}{l} 1 + \dfrac{2f(r+f+\mu)(\mu - \mu')}{(f+r)(r\mu + f\mu' + \mu\mu')} + \dfrac{f(f+\mu)(\mu - \mu')^2}{\mu(f+r)(r\mu + f\mu' + \mu\mu')} \\ + \dfrac{f(f+\mu)(f+2\mu)(\mu - \mu')^3}{\mu(f+r)(r\mu + f\mu' + \mu\mu')(r\mu + f\mu' + 2\mu\mu')} - \dfrac{\mu^2[(f\mu + r\mu') + (f+r)^2]^2}{(f+r)^2 (r\mu + f\mu' + \mu\mu')^2} \end{array} \right]. \end{aligned} \tag{3.7}$$

ii) For the case when $\mu' = 0$, the expected number of customers in equilibrium is given as

$$E(X) = \frac{\lambda}{\mu} \left(1 + \frac{f}{r} \left(1 + \frac{\mu}{r+f} \right) \right), \tag{3.8}$$

its variance is given as

$$Var(X) = \frac{\lambda}{\mu} \left(1 + \frac{f}{r} \left(1 + \frac{\lambda}{r} + \frac{\mu}{r+f} \left(1 + \frac{\lambda}{r} + \frac{\lambda}{r+f} \right) \right) \right). \tag{3.9}$$

Proof: The proof is deferred to Section 4.

Remark. From equation pairs (3.6-3.7) and (3.8-3.9), one could deduce that $Var(X) > E(X)$ in both cases.

4. Analytical Derivations

In this section, we will prove Theorem 1 and Corollary 1. We first consider the M/M/∞ queueing system described above subject to random system breakdowns of exponentially distributed durations affecting all servers.

Proof of Theorem 1(ii): For the case when $\mu' = 0$, the equations (2.3) reduce to

$$G_N'(z) = \left(\frac{\lambda}{\mu} + \frac{\lambda f}{\mu\left(-\lambda(z-1)+r\right)}\right) G_N(z), \tag{4.1}$$

$$G_F(z) = \frac{f}{(\lambda - \lambda z + r)} G_N(z). \tag{4.2}$$

Solving (4.1) yields

$$G_N(z) = Ce^{\frac{\lambda}{\mu}z}\left[\lambda(1-z)+r\right]^{-\frac{f}{\mu}}. \tag{4.3}$$

Then, from (4.2) and (4.3) and $G(z) = G_F(z) + G_N(z)$, it follows that

$$G(z) = Ce^{\frac{\lambda}{\mu}z}\left[\lambda(1-z)+r\right]^{-\frac{f}{\mu}-1}\left(\lambda - \lambda z + r + f\right), \tag{4.4}$$

where C is an unknown constant. Since $G(1) = 1$, we have $C = e^{-\frac{\lambda}{\mu}} \dfrac{r^{\frac{f}{\mu}+1}}{r+f}$. Inserting this value in (4.4) yields

$$G(z) = e^{\frac{\lambda}{\mu}(z-1)} \Im(z). \tag{4.5}$$

where

$$\Im(z) = \frac{r}{r+f}\left(\frac{r/(\lambda+r)}{1-\lambda z/(\lambda+r)}\right)^{\frac{f}{\mu}} + \frac{f}{r+f}\left(\frac{r/(\lambda+r)}{1-\lambda z/(\lambda+r)}\right)^{\frac{f}{\mu}+1}. \tag{4.6}$$

Clearly, one can identify the terms with the superscript as the generating function of the associated generalized negative binomials. Thus, equation (3.4) follows (4.6) immediately and $\Im(z)$ could be recognized as the generating function of the random variable *Y* that is given as the mixture of two independent generalized negative binomials Y_1 and Y_2 with associated *p* and *(1-p)* values. On the other hand, since the term in front of $\Im(z)$ in (4.5) is the generating function of Poisson random variable, X_φ, equation (3.3) holds and random variables X_φ and *Y* are independent of each other. □

Proof of Corollary 1 (ii): Either by taking the first and second derivatives of *G*(*z*) in (4.5) and evaluating at *z*=1, or directly from the stationary representation (3.3) and equation (3.4), the equations (3.8) and (3.9) follow after some simplifications. □

Next, we consider the M/M/∞ system with partial breakdowns, i.e., $\mu' > 0$. The proof of Theorem 1(i) is going to be achieved via two lemmas that together solve the generating function of this system.

By defining $\underline{G}(z) = \begin{bmatrix} G_N(z) \\ G_F(z) \end{bmatrix}$, the equations (2.3) can be rewritten as the following matrix differential equation with variable coefficients

$$\underline{G}'(z) = \begin{bmatrix} \frac{1}{\mu}(\lambda - \frac{f}{z-1}) & \frac{r}{(z-1)\mu} \\ \frac{f}{(z-1)\mu'} & \frac{1}{\mu'}(\lambda - \frac{r}{z-1}) \end{bmatrix} \underline{G}(z). \tag{4.7}$$

To obtain $\underline{G}(z)$, we will follow the approach used by Keilson and Servi [13], and introduce three new functions.

Let a related vector function $\underline{\hat{G}}(u)$ be defined from $G(z)$ after a change of variable u=z-1, as

$$\underline{\hat{G}}^T(u) = \underline{G}^T(u+1) = \underline{G}^T(z).$$

Note that $\hat{G}(u) = \hat{G}_F(z) + \hat{G}_N(z)$. Define a pair of scalar functions in terms of the components of $\underline{\hat{G}}(u)$,

$$\hat{G}_\mu(u) = \mu\hat{G}_N(u) + \mu'\hat{G}_F(u), \tag{4.8}$$

$$\hat{G}_\mu{}^*(u) = \mu\hat{G}_N(u) - \mu'\hat{G}_F(u). \tag{4.9}$$

The function $\hat{G}_\mu(u)$ is the cornerstone of the analysis and satisfies the *Kummer'*s differential equation (c.f. (4.27) and Slater [18]). The first step in the analysis is to identify a second-order differential equation for $\hat{G}_\mu(u)$ and a first-order differential equation for $\hat{G}_\mu^*(u)$.

Lemma 4.1. The vector differential equation (4.7) is equivalent to the following second-order ordinary differential equation for $\hat{G}_\mu(u)$ and the first-order differential equation for $\hat{G}_\mu^*(u)$,

$$\rho^*\hat{G}^*{}_\mu(u) = \hat{G}'_\mu(u) - \rho\hat{G}_\mu(u) = e^{\rho u}\frac{d}{du}\left(e^{-\rho u}\hat{G}_\mu(u)\right), \tag{4.10}$$

$$-2u\frac{d^2}{du^2}\left(e^{-\rho u}\hat{G}_\mu(u)\right) - 2\left(\frac{f}{\mu} + \frac{r}{\mu'}\right)\frac{d}{du}\left(e^{-\rho u}\hat{G}_\mu(u)\right) + \rho*\left[-2\left(\frac{f}{\mu} - \frac{r}{\mu'}\right) + 2\rho^* u\right]e^{-\rho u}\hat{G}_\mu(u) = 0, \tag{4.11}$$

where $\rho = \frac{1}{2}\left(\frac{\lambda}{\mu} + \frac{\lambda}{\mu'}\right)$, and $\rho^* = \frac{1}{2}\left(\frac{\lambda}{\mu} - \frac{\lambda}{\mu'}\right)$.

Proof: The equation (4.7) is rewritten as

$$\begin{bmatrix} \mu u & 0 \\ 0 & \mu' u \end{bmatrix} \underline{\hat{G}'}(u) = \begin{bmatrix} \lambda u - f & r \\ f & \lambda u - r \end{bmatrix} \underline{\hat{G}}(u) .$$

This implies

$$\mu u \hat{G}_N'(u) = (\lambda u - f)\hat{G}_N(u) + r\hat{G}_F(u) , \tag{4.12}$$

$$\mu' u \hat{G}_F'(u) = f\hat{G}_N(u) + (\lambda u - r)\hat{G}_F(u) . \tag{4.13}$$

Combining equations (4.8), (4.12) and (4.13) yields

$$\hat{G}_\mu'(u) = \lambda\hat{G}_N(u) + \lambda\hat{G}_F(u) . \tag{4.14}$$

Equations (4.8) and (4.9) imply

$$\hat{G}_N(u) = \frac{1}{2\mu}\left(\hat{G}_\mu(u) + \hat{G}^*{}_\mu(u)\right), \tag{4.15}$$

$$\hat{G}_F(u) = \frac{1}{2\mu'}\left(\hat{G}_\mu(u) - \hat{G}^*{}_\mu(u)\right). \tag{4.16}$$

Inserting the equations (4.15) and (4.16) into (4.14) we obtain

$$\hat{G}_\mu'(u) = \frac{1}{2}\left(\frac{\lambda}{\mu} + \frac{\lambda}{\mu'}\right)\hat{G}_\mu(u) + \frac{1}{2}\left(\frac{\lambda}{\mu} - \frac{\lambda}{\mu'}\right)\hat{G}^*{}_\mu(u) . \tag{4.17}$$

By definitions of ρ and ρ^*, equation (4.17) can be rewritten as

$$\hat{G}_\mu'(u) = \rho\hat{G}_\mu(u) + \rho^*\hat{G}^*{}_\mu(u) .$$

Moreover, since

$$\hat{G}_\mu'(u) - \rho\hat{G}_\mu(u) = e^{\rho u}\frac{d}{du}\left(e^{-\rho u}\hat{G}_\mu(u)\right) = \rho^*\hat{G}^*{}_\mu(u) ,$$

we have (4.10), i.e.,

$$\frac{d}{du}\left(e^{-\rho u}\hat{G}_{\mu}(u)\right)=e^{-\rho u}\rho^{*}\hat{G}^{*}{}_{\mu}(u)\,,$$

and

$$\frac{d^{2}}{du^{2}}\left(e^{-\rho u}\hat{G}_{\mu}(u)\right)=\rho^{*}e^{-\rho u}\frac{d}{du}\left(\hat{G}^{*}{}_{\mu}(u)\right)-\rho\rho^{*}e^{-\rho u}\hat{G}^{*}{}_{\mu}(u)\,. \tag{4.18}$$

Combining (4.9), (4.12) and (4.13), gives

$$u\left(\mu\hat{G}_{N}'(u)-\mu'\hat{G}_{F}'(u)\right)=u\frac{d}{du}\left(\hat{G}^{*}{}_{\mu}(u)\right)=(\lambda u-2f)\hat{G}_{N}(u)+(2r-\lambda u)\hat{G}_{F}(u)\,. \tag{4.19}$$

Inserting equations (4.15) and (4.16) into Eq. (4.19) yields

$$\frac{d}{du}\left(\hat{G}^{*}{}_{\mu}(u)\right)=\left[\lambda\left(\frac{1}{2\mu}-\frac{1}{2\mu'}\right)-\frac{1}{u}\left(\frac{f}{\mu}-\frac{r}{\mu'}\right)\right]\hat{G}_{\mu}(u)+\left[\lambda\left(\frac{1}{2\mu}+\frac{1}{2\mu'}\right)-\frac{1}{u}\left(\frac{f}{\mu}+\frac{r}{\mu'}\right)\right]\hat{G}^{*}{}_{\mu}(u)\,. \tag{4.20}$$

Substituting (4.20) in (4.18) yields (4.11). □

Next, with an additional transformation of variables, the o.d.e. for $\hat{G}_{\mu}(u)$ will be put into the *Kummer*'s differential equation, whose solution apart from a multiplicative constant is given in terms of *Kummer* (c.f. (4.23)) and *Tricomi* (c.f. (4.24)) functions. The solution for $\hat{G}^{*}_{\mu}(u)$ will also be given.

Lemma 4.2. The solutions to the equations (4.10) and (4.11) are given by

$$\hat{G}_{\mu}(u)=e^{\frac{\lambda}{\mu}u}\left[C_{1}M(a,b,w)+C_{2}U(a,b,w)\right], \tag{4.21}$$

and

$$\hat{G}^{*}_{\mu}(u)=e^{\frac{\lambda}{\mu}u}\left[C_{1}\left(M(a,b,w)-\frac{2a}{b}M(a+1,b+1,w)\right)+C_{2}\left(U(a,b,w)+2aU(a+1,b+1,w)\right)\right], \tag{4.22}$$

where C_1 and C_2 are two unknown constants, and $M(a, b, w)$ is the *Kummer*'s function [18] with the following power series representation when b is not an integer,

$$M(a,b,w)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{w^n}{n!}, \tag{4.23}$$

where $(a)_n = a(a+1)(a+2)\ldots(a+n-1)$, and $(a)_0 = 1$. $U(a, b, w)$ is the Tricomi's function [18] given as,

$$U(a,b,w)=\frac{\pi}{\sin \pi b}\left\{\frac{M(a,b,w)}{\Gamma(1+a-b)\Gamma(b)}-w^{1-b}\frac{M(1+a-b,2-b,w)}{\Gamma(a)\Gamma(2-b)}\right\}. \tag{4.24}$$

Proof: By letting $w=-2\rho^* u$ and $e^{-\rho u}\hat{G}_\mu(u)=e^{-w/2}f(w)$, we have

$$\frac{d}{du}\left(e^{-w/2}f(w)\right)=\rho^* e^{-w/2}f(w)-2\rho^* e^{-w/2}f'(w), \tag{4.25}$$

$$\frac{d^2}{du^2}\left(e^{-w/2}f(w)\right)=(\rho^*)^2 e^{-w/2}f(w)-4(\rho^*)^2 e^{-w/2}f'(w)+4(\rho^*)^2 e^{-w/2}f''(w). \tag{4.26}$$

Substituting Eq. (4.25) and (4.26) into (4.11) gives

$$wf''(w)+\left(\frac{f}{\mu}+\frac{r}{\mu'}-w\right)f'(w)-\frac{f}{\mu}f(w)=0. \tag{4.27}$$

This second order differential equation is called the *Kummer*'s differential. If we let $a=\frac{f}{\mu}, b=\left(\frac{f}{\mu}+\frac{r}{\mu'}\right)$, then the complete solution of (4.25) is given in Slater [18] as

$$f(w)=C_1M(a,b,w)+C_2U(a,b,w),$$

where M is the *Kummer*'s function, U is the *Tricomi*'s function, and C_1 and C_2 are two unknown constants. Furthermore, the derivative of $f(w)$ with respect to w is

$$f'(w)=C_1\frac{a}{b}M(a+1,b+1,w)-C_2aU(a+1,b+1,w).$$

Since

$$\hat{G}_\mu(u) = e^{\rho u} e^{-w/2} f(w) = e^{\frac{\lambda}{\mu}u} f(w),$$

and

$$\hat{G}^*{}_\mu(u) = \frac{e^{\rho u}}{\rho^*}\frac{d}{du}\left(e^{-w/2} f(w)\right) = e^{\frac{\lambda}{\mu}u}\left(f(w) - 2f'(w)\right),$$

the equations (4.21) and (4.22) follow immediately. □

Before we prove Theorem 1(i), we show that the generating function of Poisson random variable *Y* randomized by truncated beta *ℬ(a,b,c),* first introduced in Section 3 via its probability mass function (3.2), is the *Kummer*'s function *M(a,b,c(z-1))*. We include this derivation for the purpose of completeness; see also Fitzgerald [6].

We can evaluate the generating function of *Y* directly as

$$\psi(z) = \sum_{k=0}^{\infty} P\{Y = k\} z^k = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_{\gamma=0}^{c} e^{\gamma(z-1)} (\gamma / c)^{a-1} (1 - \gamma / c)^{b-a-1} d\gamma / c .$$

Let $y = \gamma / c$, then $\psi(z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_{y=0}^{1} e^{cy(z-1)} y^{a-1} \left(1-y\right)^{b-a-1} dy$. Solution to this integral is given in Slater [18] in terms of the *Kummer*'s function, i.e., $\frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)} M\left(a,b,c(z-1)\right)$.

Thus,

$$\psi(z) = M(a, b, c(z\text{-}1)). \tag{4.28}$$

Finally, we present the proof of Theorem 1(i).

Proof of Theorem 1(i): Since $\hat{G}(u) = \hat{G}_F(z) + \hat{G}_N(z)$, equations (4.15) and (4.16) give

$$\hat{G}(u)=\left(\frac{1}{2\mu}+\frac{1}{2\mu'}\right)\hat{G}_{\mu}(u)+\left(\frac{1}{2\mu}-\frac{1}{2\mu'}\right)\hat{G}^{*}{}_{\mu}(u)=\frac{1}{\lambda}\left(\rho\hat{G}_{\mu}(u)+\rho^{*}\hat{G}^{*}{}_{\mu}(u)\right). \tag{4.29}$$

Using the conclusion in Lemma 4.2 yields

$$\hat{G}(u)=e^{\frac{\lambda}{\mu}u}\left\{C_1\left(\frac{1}{\mu}M(a,b,-2\rho^*u)-\frac{2a\rho^*}{b\lambda}M(a+1,b+1,-2\rho^*u)\right)+C_2\left(\frac{1}{\mu}U(a,b,-2\rho^*u)+\frac{2a\rho^*}{\lambda}U(a+1,b+1,-2\rho^*u)\right)\right\}.$$

Since $G(z)\big|_{z=1}=1$ and $u = z+1$, $\hat{G}(u)\big|_{u=0}=1$. But, $U(a+1,b+1,w)\xrightarrow{w\to 0}\infty$ when b > 0. Thus, C_2 must be 0. Moreover, since $M(a, b, 0) = 1$, C_1 is obtained as $C_1=\dfrac{r\mu+f\mu'}{r+f}$.

Finally, from (4.29), the generating function of this system is written as

$$G(z)=\frac{r\mu+f\mu'}{r+f}e^{\frac{\lambda}{\mu}(z-1)}\left(\frac{1}{\mu}M(a,b,-2\rho^*(z-1))-\frac{2a\rho^*}{b\lambda}M(a+1,b+1,-2\rho^*(z-1))\right). \tag{4.30}$$

Using the relations in (3.5), equation (4.30) could be rewritten as

$$G(z)=e^{\frac{\lambda}{\mu}(z-1)}\Psi(z), \tag{4.31}$$

where,

$$\Psi(z)=\left[\frac{(r\mu+f\mu')}{(r+f)\mu}M(a,b,-2\rho^*(z-1))+\frac{f(\mu-\mu')}{(r+f)\mu}M(a+1,b+1,-2\rho^*(z-1))\right]. \tag{4.32}$$

By equation (4.28), one can identify the *Kummer*'s functions $M(a,b,-2\rho^*(z-1))$ and $M(a+1,b+1,-2\rho^*(z-1))$ in (4.32) as the generating functions of Y_1 and Y_2, respectively. Here, note that $c=-2\rho^*=\dfrac{\lambda}{\mu'}-\dfrac{\lambda}{\mu}>0$. Thus, equation (3.4) follows (4.32) immediately and $\Psi(z)$ could be recognized as the generating function of *Y* that is given as the mixture of random variables Y_1 and Y_2 with associated *p* and *(1-p)* values. Clearly, since the term in front of $\Psi(z)$ in (4.31) is the generating function of Poisson random variable X_φ, equation (3.3) holds and random variables X_φ and *Y* are independent of each other. □

Proof of Corollary 1(i): By taking the derivative of $G(z)$ and evaluating at z=1, or directly from the representation of Eq. (4.23) and (4.24), and also using the formulas in Slater [18], the expected number of customers in the system at steady state can be calculated as

$$E(X)=\frac{\lambda}{\mu}+\left[p\frac{a(-2\rho^{*})}{b}+(1-p)\frac{(a+1)(-2\rho^{*})}{b+1}\right], \tag{4.33}$$

its variance as

$$\begin{aligned}Var(X)=\frac{\lambda}{\mu}+p\left[\frac{a(-2\rho^{*})}{b}+\frac{a(b-a)(-2\rho^{*})^{2}}{b^{2}(b+1)}\right]+(1-p)\left[\frac{(a+1)(-2\rho^{*})}{b+1}+\frac{(a+1)(b-a)(-2\rho^{*})^{2}}{(b+1)^{2}(b+2)}\right]\\ +p(1-p)\frac{(b-a)^{2}(-2\rho^{*})^{2}}{b^{2}(b+1)^{2}}\ .\end{aligned} \tag{4.34}$$

Replacing the p, a, b and ρ^{*} with their respective definitions yields equations (3.6) and (3.7). □

5. Discussion and future research

In this paper, we present the closed-form solution to the M/M/∞ queueing system subject to random interruptions of exponentially distributed durations. Under the impact of the interruption, all servers work at lower efficiency until the interruption is cleared. We give the complete representation of the number of customers in the system in equilibrium. Equation (3.6) could be used to see the impact of each parameter on the expected number of customers in the system in equilibrium. Figure 5.1 illustrates the situations with minor interruptions where interruptions cause the service rate to drop to one third of the normal service rate. Figure 5.2 shows the situations with serious interruptions, which reduce the service rate 90%. All curves in both figures illustrate that the number of customers in the system decreases while the service rate increases. By comparing Figure 5.1 with Figure 5.2, we conclude that, serious interruptions cause the values of f and r to have more significant effect on the expected number of customers in the system than the minor interruption cases. In both figures, higher f values result in more customers in the system. The effect of f is illustrated by a curve with much higher f value.

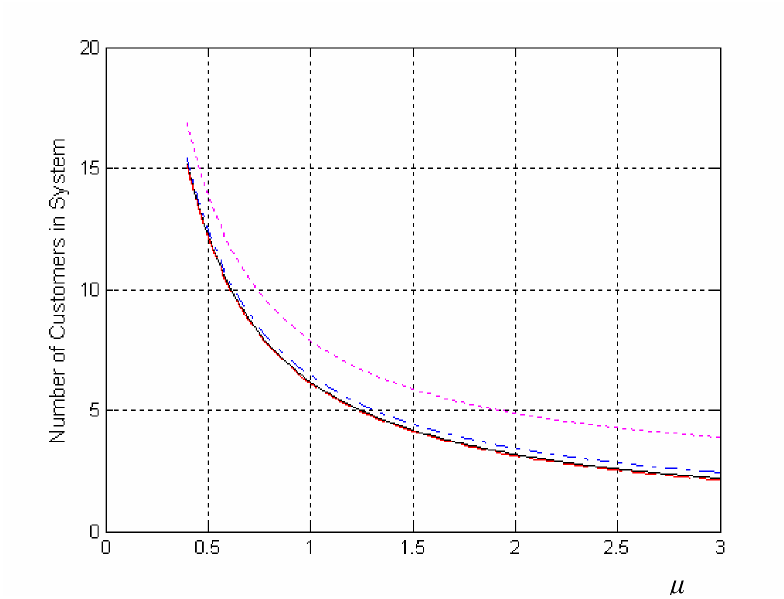

Figure 5.1 Expected number of customers in the system with $\mu=3\mu'$

(-.- f=0.005, r=0.05; — f=0.002, r=0.05; -- f=0.002, r = 0.075; … f=0.05, r = 0.075)

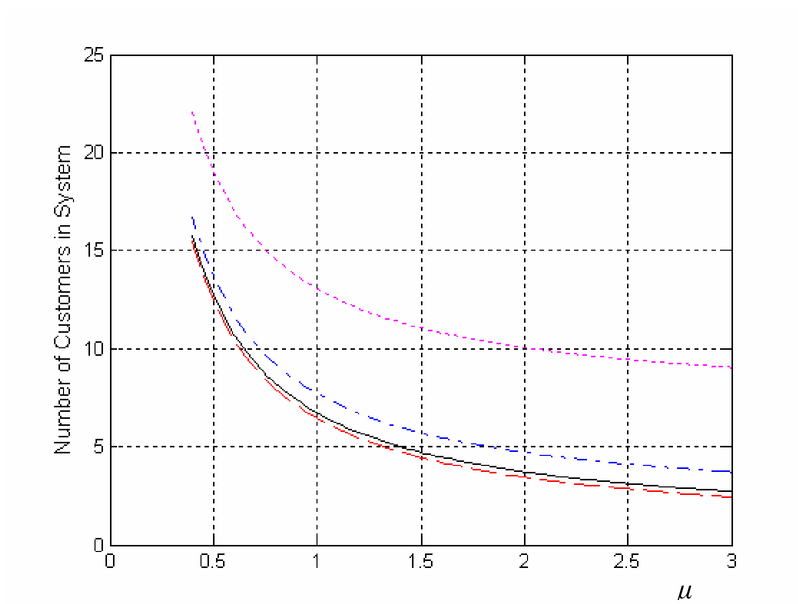

Figure 5.2 Expected number of customers in the system with $\mu=10\mu'$

(-.- f=0.005, r=0.05; — f=0.002, r=0.05; -- f=0.002, r = 0.075; … f=0.05, r = 0.075)

There are many potential applications of the queueing model discussed in this paper from transportation to telecommunication. Another application of this model is in the modern working environment. For example, in most of the current library systems, the procedure of check-in and checkout of books is performed by scanning the barcode, which is efficient and convenient. But during the routine maintenance of the computer system or power failure, the librarians have to do these manually. This model could be used to evaluate the impact of the maintenance or estimate how many more employees should be hired to maintain the quality of service during those special situations.

In this paper, we only consider interruptions at the same severity level. A multi-state queueing model is a direct extension to this study. In practice, a system is often subject to interruptions with different severities. Each type of interruptions has its own frequency of occurrence, duration and repair time, and their negative impacts on the system could also be different. Another extension is to consider the queueing models with finite number of servers. If this is the case, the system service rate increase proportionally with the number of vehicles in the system only before the number of vehicles reaches the maximum number of available servers. Afterward, the system service rate remains stable. However, if the number of servers is high enough compared to the arrival rate of customers, like the road segment mentioned in the Introduction, our model could be used to obtain an approximate solution.

References

[1] ALTIOK, T., Queues with group arrivals and exhaustive service discipline, *Queueing Systems*, Vol. 2, (1987), 307-320.

[2] AVI-ITZHAK, B. AND NAOR, P, Some Queueing Problems with the Service Station subject to breakdown, *Operations Research*, Vol. 11, Issue 3, (1963), 303-320.

[3] CHAO, X. AND ZHAO, Y.Q., Analysis of multi-server queues with station and server vacations, *European Journal of Operational Research*, 110, Issue 2, (October 1998), 392-406.

[4] COOPER, R. B., Queues served in cyclic order: Waiting times, *Bell Syst. Tech. J.*, 49, (1970), 399-413.

[5] DOSHI, B. T., Single Server Queues with Vacations, in *Stochastic Analysis of Computer and Communication Systems* edited by H. Takagi, (1990), 217-265.

[6] FITZGERALD, D. L., Tricomi and Kummer Functions in Occurrence, Waiting Time and Exceedance Statistics, *Stochastic Environmental Research and Risk Analysis*, 16, (2002), 207-224.

[7] FUHRMANN, S. W., A note on the M/G/1 queue with server vacations, *Op. Res.*, Vol. 32 (1984) 1368-1373.

[8] FUHRMANN, S. W. AND COOPER, R.B., Stochastic decomposition in the M/G/1 queue with generalized vacations, *Op. Res.*, Vol. 33, (1985) 1117-1129.

[9] GAVER, D. P. Jr., A waiting line with interrupted service, including priorities, *J. Roy. Stat. Soc*. B24, (1962), 73-90.

[10] HARRIS, C. M. AND MARCHAL, W. G., State dependence in M/G/1 server-vacation models, *Op. Res.*, Vol. 36, No. 4, July-August (1988) 560-565.

[11] JAIN, R. AND SMITH, J. M., Modeling Vehicular traffic flow using M/G/C/C state dependent queueing models, *Transportation Science*, 31, No. 4 (1997) 324-336.

[12] KEILSON, J., Queues subject to service interruptions, *Ann. Math. Statistics*, 33, (1962), 1314-1322.

[13] KEILSON, J. AND SERVI, L.D., The matrix *M/M/∞* system: Retrial models and Markov modulated sources, *Advances in Applied Probability*, vol. 25 (1993) 453-471.

[14] LEVY, Y. AND YECHIALI, U., Utilization of idle time in an M/G/1 queueing system, *Management Sc*., 22, (1975), 202-211.

[15] MITRANY, I. L. AND AVI-ITZHAK, B., A many-server queue with service interruptions, *Op. Res*., Vol. 16, No.3, May-June (1968) 628-638.

[16] NEUTS, M. F., *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach,* The John Hopkins University Press, 1981.

[17] SHANTHIKUMAR, J. G., On stochastic decomposition in M/G/1 type queues with generalized server vacations, *Op. Res*., Vol. 36 (1986) 566-569.

[18] SLATER, L. J., Confluent Hypergeometric Functions, 1960, CUP, Cambridge, UK, also in *Handbook of Mathematical Functions,* edited by M. Abromowitz and I. Stegun (1964) 503-515.

[19] WHITE, H. C. AND CHRISTE, L. S., Queueing with preemptive priorities or with breakdown, *Op. Res*., 6, (1958) 79-95.

[20] YADIN, M. AND NAOR, P., Queueing systems with a removable service station, *Op. Res. Quart*., 14 (1963) 393-405.

[21] ZHANG, Z. G. AND TIAN, N., Analysis on queueing systems with synchronous vacations of partial servers, *Performance Evaluation*, 52, Issue 4, May (2003) 269-282.