Inventory Control under Substitutable Demand: A Stochastic Game Application

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Abstract

Substitutable product inventory problem is analyzed using the concepts of stochastic game theory. It is assumed that there are two substitutable products that are sold by different retailers and the demand for each product is random. Game theoretic nature of this problem is the result of substitution between products. Since – retailers compete for the substitutable demand, ordering decision of each retailer depends on the ordering decision of the other retailer. Under the discounted payoff criterion, this problem is formulated as a two-person nonzero-sum stochastic game. In the case of linear ordering cost, it is shown that there exists a Nash equilibrium characterized by a pair of stationary base stock strategies for the infinite horizon problem and this is the unique Nash equilibrium within the class of stationary base stock strategies.

1 Introduction

This study focuses on investigating the equilibrium strategies for substitutable product inventory control systems within the class of stationary base stock strategies. When different products are sold by different retailers, substitution between these products causes the retailers to decide on their order quantities in a competitive environment, and thus, the game theoretic nature of the problem arises. In this article, a nonzero-sum discounted stochastic game formulation is given for the two-product problem. The retailers observe their inventory levels periodically and take actions according to their ordering policies. It is assumed that both retailers behave rationally, i.e., each retailer tries to optimize his own payoff. The set-up cost of each retailer is assumed to be zero.

Substitutable product inventory problem was first studied by McGillivray and Silver [7] in the Economic Order Quantity(EOQ) context. Later, Parlar and Goyal [10] and Khouja and Mehrez and Rabinowitz [5] gave single-period formulations for an inventory system with two substitutable products independently of each other. In [8], Parlar proposed a Markov Decision Process model to find the optimal ordering policies for perishable and substitutable products from the point of view of one retailer. Parlar's study in [9] is a game theoretic analysis of the inventory control under substitutable demand. He modeled the two-product single-period problem as a two-person nonzero-sum game and showed that there exists a unique Nash equilibrium.

As an extension of the model in [9], Wang and Parlar [15] studied the three-product single-period problem.

In this study, the work in [9] is extended -to- the infinite horizon and lost sale case. The solution of the corresponding nonzero-sum stochastic game is considered over the class of stationary base stock strategies because this makes both implementation of the strategies and analysis of the problem easier. It is shown that under the discounted payoff criterion – retailers possess a stationary base stock Nash strategy pair which is the unique Nash equilibrium within the class of stationary base stock strategies. Stationary base stock strategies are expressed by constant order-upto-levels. If the inventory at the beginning of a period is below -the- order-upto-level, then an order is given to bring inventory amount to that level; otherwise, no action is taken. Also, it is observed that cooperation of – retailers leads to a better total payoff than the sum of the individual payoff amounts of the retailers in the non-cooperative case.

There are two other models that are related to the substitutable product inventory model. In [14], Veinott considered a single retailer inventory problem under backlogging, budget and/or capacity limitations. He gave conditions to ensure that the base stock ordering policy is optimal for the expected discounted cost criterion. Later, Ignall and Veinott [4] considered the same model and obtained new conditions under which a myopic ordering policy (a policy of minimizing the expected cost in one period only) is optimal for a sequence of periods. An important one of these conditions is the so-called substitute property. This property holds when the myopic policy is such that increasing the initial inventory of any product does not increase the stock on hand after ordering of any other product. This property arises in the models that include some kind of product substitution such as substituting storage space for one product for that of another, or the demand for a product at one location for that at another location in a multi location inventory model. But, the substitution between products in the sense of this paper destroys the optimality of the myopic policy. Since there is only one retailer, there is no competition in this model. In [6], Kirman and Sobel considered a dynamic oligopoly model with inventories. In oligopolies, a small number of firms produce homogeneous or comparable goods competitively. –Firms compete by increasing the demand for their product via advertisement, pricing or by keeping inventories. They considered the case of backlogging and discounting, and analyzed this model using the stochastic game approach. For the infinite horizon case, they gave conditions under which the game has a Nash equilibrium and each firm has a base stock type myopic policy. One condition requires that the demand function is smooth. This condition eliminates the cases such as all customers always choose to buy from the firm with the lowest price. Although there is competition in this model, the substitution between products is not considered.

Organization of this article is as follows: The problem and the notation are introduced, and the model is developed in section 2. Then, in section 3, analyses are presented for the use of stationary base stock strategies and cooperation of the retailers is discussed.

2 Model of the Substitutable Product Inventory Problem

In the analysis of the substitutable product inventory problem over infinite horizon, concepts of nonzero-sum stochastic games are used. Two retailers of different products who compete for the substitutable demand of these products are the players of the game. Demand distributions of the products and the substitution rates are known by both players. For the nonzero-sum stochastic game formulation, Nash equilibrium is considered. Unilateral deviations of either of the players from his Nash strategy do not improve his expected payoff.

Retailers observe their inventory levels at the beginning of each period and make their ordering decisions accordingly. A period is named (indexed) by the number of periods from the beginning of that period until the end of the planning horizon, i.e., period n means that there are n decision epochs to go until the end of the planning horizon. Let X_n and Y_n be the independently and identically distributed(i.i.d) random variables denoting the demand -for- product 1 and 2, respectively, in period n. Product i is sold for q_i

per unit, $i = 1, 2$. Order-ing- cost is a linear function of the order quantity Q_{in} for product i, $i = 1, 2$, in period n. c_i, that satisfies $0 < c_i < q_i$, is the order0ing- cost per unit of product i, $i = 1, 2$. Orders are delivered instantaneously. l_i is the unit lost sale cost and h_i is the inventory holding cost per unit of product i per period. –Substitution rates are given as the probabilities that a customer switches from one type of product to the other when the product demanded is sold out. a (b) is the probability that a customer of product 1 (2) switches to product 2 (1) given that product 1 (2) is sold out. Let I_n and J_n be the inventory levels of retailers I and II, respectively, at the beginning of period n. At each epoch n, (I_n, J_n) denotes the state of the stochastic process and (Q_{1n}, Q_{2n}) denotes the action pair taken by the retailers. Then, $z_{1n} = I_n + Q_{1n}$ and $z_{2n} = J_n + Q_{2n}$ are the inventory levels just after the orders are replenished. –The backward dynamic equations associated with the state variables are given as: $I_{n-1} = [z_{1n} - x_n - b[y_n - z_{2n}]^+]^+$ and $J_{n-1} = [z_{2n} - y_n - a[x_n - z_{1n}]^+]^+$ where $[a]^+ = max\,\{0, a\}$. Note that if retailer II cannot satisfy demand Y_n fully, then the remaining demand $(Y_n - z_{2n})^+$ is satisfied by Retailer I and vice versa. Let $P^1_{(I,J)(z_1-I)(z_2-J)}$ be the one-period expected payoff for the first player in state (I, J) when the order quantities are Q_1 and Q_2, i.e., when the order-upto-levels are $z_1 = I + Q_1$ and $z_2 = J + Q_2$. By suppressing the subscript n, the one-period expected payoff has the following form:

$$p^1_{(I,J)(z_1-I)(z_2-J)} = l_1 E[(X - z_1)^+] + c_1(z_1 - I) + h_1 E[z_1 - X - b(Y - z_2)^+] - q_1 E[\min X, z_1 + \min b(Y - z_2)^+, (z_1 - X)^+].$$

The second retailer's expected payoff is defined similarly.

For the sake of simplicity of the analysis, assume that the nonnegative random demand variables X and Y have continuous density functions f and g, respectively, with finite expectations. Let $f(0) = 0$, $g(0) = 0$. The corresponding cumulative and complementary cumulative functions will be denoted by F, G and $\bar{F}$, $\bar{G}$, respectively. One-period expected payoff $p^1_{(I,J)(z_1-I)(z_2-J)}$ has the following form:

$$\begin{aligned}
p^1_{(I,J)(z_1-I)(z_2-J)} \quad = \quad & l_1 E(X) - (q_1 + l_1) \int_0^{z_1} x f(x) dx - (q_1 + l_1) z_1 \int_{z_1}^{\infty} f(x) dx \\
& + h_1 \int_0^{z_1} (z_1 - x)\, f(x) dx + c_1(z_1 - I) \\
& - (q_1 + h_1) \int_0^{z_1} \int_{z_2}^{z_2 + \frac{z_1 - x}{b}} b(y - z_2)\, g(y) f(x) dy dx \\
& - (q_1 + h_1) \int_0^{z_1} \int_{z_2 + \frac{z_1 - x}{b}}^{\infty} (z_1 - x)\, g(y) f(x) dy dx.
\end{aligned}$$

Parlar [9] investigated this problem for the single-period case. He analyzed the -reward- function $-c^1$ and showed that it is concave in z_1. He also proved that there exists a unique Nash equilibrium specified with order-upto-levels, say S_1 and S_2 for retailers I and II, respectively.

For the discrete demand case, it is possible to numerically solve the single-period problem although the size of the state space may make it impractical. Under the long-run average payoff criterion, the nonlinear programming formulation developed by Filar et.al. [2] can be used to compute Nash strategies. If the discounted payoff criterion is considered, then NLP due to Raghavan and Filar [11] is available.

3 Stationary Base Stock Nash Strategies

The purpose of this section is to investigate Nash equilibrium of the infinite horizon substitutable product inventory problem within the class of stationary base stock strategies. To this end, first the finite horizon problem is analyzed from the viewpoint of retailer I by assigning a stationary base stock strategy to retailer II. The results obtained are then extended for the infinite horizon problem and it is observed that when retailer II uses a stationary base stock strategy the optimal strategy of retailer I is also a stationary base stock strategy. Finally, existence and uniqueness of a Nash solution within the class of stationary base stock strategies are proved.

In the multi-period model analyzed in this article, each retailer tries to minimize his expected payoff. Define $L(z_1, z_2)$ as one-period expected payoff except the ordering cost $P_1(z_1 - I)$ as follows:

$$\begin{aligned} L_1(z_1, z_2) &= p^1_{(I,J)(z_1-I)(z_2-J)} - c_1(z_1 - I) \\ &= l_1 \int_0^{z_1} (z_1 - x) f(x) dx - (q_1 + l_1) z_1 + l_1 E(X) \\ &\quad + (q_1 + h_1) \int_0^{z_1} \int_0^{z_2 + \frac{z_1 - x}{b}} (z_1 - x - b[y - z_2]^+) g(y) f(x) dy dx. \end{aligned}$$

Let $C_{1n}(I, J)$ represent the minimum expected discounted payoff of retailer I for the remaining n periods until the end of the planning horizon given that the beginning inventory I_n (J_n) of retailer I (II) is I (J) and the inventory level of product 1 (2) is z_1 (z_2) just after the replenishment. For player II, $C_{2n}(I, J)$ is defined similarly. The discount factor is assumed stationary and will be denoted by γ, $0 < \gamma < 1$. $C_{1n}(I, J)$ satisfies the following functional equation:

$$\begin{aligned} C_{1n}(I, J) = \quad & \min\nolimits_{z_1 \geq I} \; c_1(z_1 - I) + L_1(z_1, z_2) \\ & + \gamma \textstyle\int_0^\infty \int_0^\infty C_{1(n-1)}([z_1 - x - b[y - z_2]^+]^+, [z_2 - y - a[x - z_1]^+]^+) \, g(y) f(x) dy dx, \end{aligned}$$

where the first two terms at the right hand side correspond to the one-period expected payoff. Here, optimal action of retailer I (the minimizing value of z_1 in the above equation) is determined for a given order-upto-level z_2 of retailer II. Note that the function that is minimized has a constant part, i.e., $-c_1 I$, and a variable part, say $D_{1n}(z_1, z_2)$. Thus,

$$C_{1n}(I, J) = \min_{z_1 \geq I} \; D_{1n}(z_1, z_2) - c_1 I,$$

and the minimization is performed only on $D_{1n}(z_1, z_2)$.

Consider the finite horizon problem for retailer I when a stationary base stock strategy is employed by retailer II. Relative to the initial inventories of product 2 less than or equal to z_2, retailer II starts every period with z_2 units of product 2. As in Scarf[12], the results presented in this section are obtained by an inductive analysis of $D_{1n}(z_1, z_2)$:

$$\begin{aligned} D_{1n}(z_1, z_2) = \quad & c_1 z_1 + L_1(z_1, z_2) \\ & + \gamma \textstyle\int_0^\infty \int_0^\infty C_{1(n-1)}([z_1 - x - b[y - z_2]^+]^+, [z_2 - y - a[x - z_1]^+]^+) \, g(y) f(x) dy dx, \end{aligned}$$

for every n. Considering the value of I_{n-1} for different pairs of demand values, $D_{1n}(z_1, z_2)$ may be further decomposed as follows:

$$\begin{aligned} D_{1n}(z_1, z_2) &= c_1 z_1 + L_1(z_1, z_2) + \gamma G(z_2) \int_0^{z_1} C_{1(n-1)} \left(z_1 - x, z_2 \right) f(x) dx \\ &\quad + \gamma \int_0^{z_1} \int_{z_2}^{z_2 + \frac{z_1 - x}{b}} C_{1(n-1)}((z_1 - x - b(y - z_2), z_2) \, g(y) f(x) dy dx \\ &\quad + \gamma C_{1(n-1)}(0, z_2) \left(\int_0^{z_1} \int_{z_2 + \frac{z_1 - x}{b}}^\infty g(y) f(x) dy dx + \bar{F}(z_1) \right). \end{aligned}$$

If $D_{1n}(z_1, z_2)$ is convex in z_1 for a given order-upto-level z_2 of retailer II, then optimal strategy of retailer I is a base stock strategy. Order-upto-level of this strategy is the minimizing point of $D_{1n}(z_1, z_2)$, which will be denoted by S_{1n}. Note that S_{1n} is a function of z_2. Lemma shows that for a given z_2 in $[0, \infty)$, $D_{1n}(z_1, z_2)$ is convex in z_1 and the minimizing point S_{1n} is greater than zero. Note that $S_{1n} > 0$ because $\lim_{z_1 \to 0} \frac{\partial}{\partial z_1} D_{1n}(z_1, z_2) < 0$ for every n.

Lemma 1 *If retailer II uses a stationary base stock strategy with order-upto-level z_2, then for $n = 1, 2, ...$*
(i) *$D_{1n}(z_1, z_2)$ is convex in z_1,*
(ii) $\lim_{z_1 \to 0} \frac{\partial}{\partial z_1} D_{1n}(z_1, z_2) < 0$.

Proof: The proof is given by induction on the number of periods remaining. For period $n = 1$, $D_{11}(z_1, z_2) = c_1 z_1 + L_1(z_1, z_2)$ holds by taking $C_{10} = 0$. The partial derivative of $D_{11}(z_1, z_2)$ is given using the Leibnitz's rule of differentiation:

$$\begin{aligned}\frac{\partial}{\partial z_1} D_{11}(z_1, z_2) = \quad & c_1 - (q_1 + l_1) \int_{z_1}^{\infty} f(x)dx + h_1 \int_0^{z_1} f(x)dx \\ & -(q_1 + h_1) \int_0^{z_1} \int_{z_2 + \frac{z_1 - x}{b}}^{\infty} g(y) f(x) dy dx.\end{aligned}$$

Then,

$$\begin{aligned}\lim_{z_1 \to 0} \frac{\partial}{\partial z_1} D_{11}(z_1, z_2) &= c_1 - (q_1 + l_1) \int_0^{\infty} f(x)dx \\ &= -(q_1 + l_1 - c_1) < 0,\end{aligned}$$

since $q_1 > c_1$, and the proof of (ii) is complete. The second partial derivative of $D_{11}(z_1, z_2)$ is

$$\frac{\partial^2}{\partial z_1^2} D_{11}(z_1, z_2) = l_1 f(z_1) + (q_1 + h_1) \left(\int_0^{z_1} g(z_2 + \frac{z_1 - x}{b}) \frac{f(x)}{b} dx + f(z_1) G(z_2) \right) \geq 0,$$

which proves (i).

Assume that the lemma is true for periods $2, 3, ..., n$. By the induction assumption, the optimal strategy of retailer I in period n is to order upto S_{1n} if the inventory level is below S_{1n} and not to order if it is above S_{1n}. Hence,

$$C_{1n}(I, z_2) = \begin{cases} c_1(S_{1n} - I) + C_{1n}(S_{1n}, z_2) & \\ = -c_1 I + D_{1n}(S_{1n}, z_2) & if\ I < S_{1n}, \\ & \\ -c_1 I + D_{1n}(I, z_2) & if\ I \geq S_{1n}. \end{cases} \tag{1}$$

To show that the lemma is true for period $(n+1)$, $D_{1(n+1)}(z_1, z_2)$ is rewritten below using the value of C_{1n} in (1) by comparing S_{1n} and the inventory level at the beginning of period n, I_n. Note that I_n may take values below or above S_{1n}. This is because z_1 may be greater than or equal to S_{1n} and the demand in period $(n+1)$ determines I_n. For $z_1 > S_{1n}$, the following inequalities are used in writing $D_{1(n+1)}(z_1, z_2)$:

$$\begin{aligned} I_n = z_1 - x < S_{1n} \qquad & if\ \ z_1 - S_{1n} < x \leq z_1,\ 0 \leq y < z_2, \\ I_n = z_1 - x \geq S_{1n} \qquad & if\ \ 0 \leq x \leq z_1 - S_{1n},\ 0 \leq y < z_2, \end{aligned}$$

$$\begin{aligned} I_n = (z_1 - x) - b(y - z_2) < S_{1n} \qquad & if\ \ z_1 - S_{1n} < x < z_1,\ \ z_2 \leq y \leq z_2 + \tfrac{z_1 - x}{b} \\ & or\ \ 0 \leq x \leq z_1 - S_{1n},\ \ z_2 + \tfrac{z_1 - x - S_{1n}}{b} \leq y \leq z_2 + \tfrac{z_1 - x}{b}, \\ I_n = (z_1 - x) - b(y - z_2) \geq S_{1n} \qquad & if\ \ 0 \leq x \leq z_1 - S_{1n},\ \ z_2 \leq y \leq z_2 + \tfrac{z_1 - x - S_{1n}}{b}. \end{aligned}$$

Now, using (1) for each of the ranges of x and y above, $D_{1(n+1)}(z_1, z_2)$ is written as follows:

$$\begin{aligned} D_{1(n+1)}(z_1, z_2) = \ & c_1 z_1 + L_1(z_1, z_2) + \gamma D_{1n}(S_{1n}, z_2) \\ & -\gamma c_1 \int_0^{z_1} \int_0^{z_2 + \frac{z_1 - x}{b}} \left(z_1 - x - b[y - z_2]^+ \right) g(y) f(x) dy dx \\ & +\gamma \int_0^{[z_1 - S_{1n}]^+} \int_0^{z_2 + \frac{z_1 - x - S_{1n}}{b}} (D_{1n}(z_1 - x - b[y - z_2]^+, z_2) \\ & \qquad -D_{1n}(S_{1n}, z_2))\ g(y) f(x) dy dx. \end{aligned}$$

When the fourth term above is combined with L_1, $D_{1(n+1)}(z_1, z_2)$ takes the following form:

$$\begin{aligned}
D_{1(n+1)}(z_1, z_2) \;\; = \;\; & c_1 z_1 + l_1 \int_0^{z_1} (z_1 - x) f(x) dx - (q_1 + l_1) z_1 + l_1 E(X) + \gamma D_{1n}(S_{1n}, z_2) \\
& + (q_1 + h_1 - \gamma c_1) \int_0^{z_1} \int_0^{z_2 + \frac{z_1 - x}{b}} \left(z_1 - x - b[y - z_2]^+ \right) g(y) f(x) dy dx \\
& + \gamma \int_0^{[z_1 - S_{1n}]^+} \int_0^{z_2 + \frac{z_1 - x - S_{1n}}{b}} (D_{1n}(z_1 - x - b[y - z_2]^+, z_2) \\
& \qquad - D_{1n}(S_{1n}, z_2)) \; g(y) f(x) dy dx. \qquad (2)
\end{aligned}$$

In order to prove (ii), take the first partial derivative with respect to z_1.

$$\begin{aligned}
\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2) \;\; = \;\; & A_1(z_1, z_2) \\
& + \gamma \int_0^{[z_1 - S_{1n}]^+} \int_0^{z_2 + \frac{z_1 - x - S_{1n}}{b}} \frac{\partial}{\partial z_1} D_{1n}(z_1 - x - b[y - z_2]^+, z_2) \; g(y) f(x) dy dx, \quad (3)
\end{aligned}$$

where $A_1(z_1, z_2)$ is the derivative of the first six terms of $D_{1(n+1)}(z_1, z_2)$ in (2), i.e.,

$$\begin{aligned}
A_1(z_1, z_2) \;\; = \;\; & c_1 - (q_1 + l_1) \int_{z_1}^{\infty} f(x) dx + (h_1 - \gamma c_1) \int_0^{z_1} f(x) dx \\
& - (q_1 + h_1 - \gamma c_1) \int_0^{z_1} \int_{z_2 + \frac{z_1 - x}{b}}^{\infty} g(y) f(x) dy dx.
\end{aligned}$$

Since $S_{1n} > 0$ by the induction assumption, the second term of $\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2)$ vanishes at $z_1 = 0$. $D_{1(n+1)}(z_1, z_2)$ is decreasing at small $z_1 \geq 0$ because the first partial derivative takes a negative value when z_1 goes to zero as shown below:

$$\lim_{z_1 \to 0} \frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2) \; = \; c_1 - (q_1 + l_1) \int_0^{\infty} f(x) dx \; = \; -(q_1 + l_1 - c_1) < 0.$$

To show (i), the second partial derivative with respect to z_1 is analyzed. At any $z_1 < S_{1n}$,

$$\begin{aligned}
\frac{\partial^2}{\partial z_1^2} D_{1(n+1)}(z_1, z_2) \;\; = \;\; & l_1 f(z_1) + (q_1 + h_1 - \gamma c_1) f(z_1) G(z_2) \\
& + (q_1 + h_1 - \gamma c_1) \int_0^{z_1} g(z_2 + \frac{z_1 - x}{b}) \; \frac{f(x)}{b} dx,
\end{aligned}$$

and it is nonnegative since $q_1 > c_1$. At $z_1 \geq S_{1n}$,

$$\begin{aligned}
\frac{\partial^2}{\partial z_1^2} D_{1(n+1)}(z_1, z_2) \;\; = \;\; & l_1 f(z_1) + (q_1 + h_1 - \gamma c_1) f(z_1) G(z_2) \\
& + (q_1 + h_1 - \gamma c_1) \int_0^{z_1} g(z_2 + \frac{z_1 - x}{b}) \frac{f(x)}{b} dx \\
& + \gamma(\; G(z_2) f(z_1 - S_{1n}) \; + \; \int_0^{z_1 - S_{1n}} g(z_2 + \frac{z_1 - x - S_{1n}}{b}) \frac{f(x)}{b} dx \;) \frac{\partial}{\partial z_1} D_{1n}(z_1, z_2)|_{z_1 = S_{1n}} \\
& + \gamma \int_0^{z_1 - S_{1n}} \int_0^{z_2 + \frac{z_1 - x - S_{1n}}{b}} \frac{\partial^2}{\partial z_1^2} D_{1n}(z_1 - x - b[y - z_2]^+, z_2) \; g(y) f(x) dy dx,
\end{aligned}$$

where the last term is nonnegative since, by the induction assumption, $D_{1n}(z_1 + h, z_2)$ is convex in z_1 for any h. The first three terms are also nonnegative. $\frac{\partial}{\partial z_1} D_{1n}(z_1, z_2)|_{z_1 = S_{1n}}$ is either zero with a finite S_{1n} value or negative with infinite S_{1n}. If S_{1n} is finite, then the fourth term is zero. Otherwise, the only case that needs to be analyzed is the first case where $z_1 < S_{1n}$. $\square$

Remark 1: **(i)** *The function, defined by the first six terms of $D_{1(n+1)}(z_1, z_2)$ in (2), is convex in z_1.*

The derivative of this function is exactly the same as $\partial D_{11}(z_1, z_2)/\partial z_1$ except that every h_1 is replaced with $(h_1 - \gamma c_1)$. The second partial derivative of this function is

$$\frac{\partial A_1(z_1, z_2)}{\partial z_1} = l_1 f(z_1) + (q_1 + h_1 - \gamma c_1)\left(\int_0^{z_1} g(z_2 + \frac{z_1 - x}{b})\frac{f(x)}{b}dx + f(z_1)G(z_2)\right),$$

which is nonnegative at $z_1 \geq 0$ for any given $z_2 \geq 0$. $\square$

(ii)$D_{1(n+1)}(z_1, z_2)$ *and* $\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2)$ *are continuous at* $z_1 = S_{1n}$.

Next, it is shown that for any given nonnegative z_2, $D_{1n}(z_1, z_2)$ attains its minimum at a finite S_{1n}. If $D_{1n}(z_1, z_2)$ has multiple minima, then S_{1n} is the smallest z_1 value at which the minimum is attained. It is proved that the sequence $\{S_{1n}\}_{n=1}^{\infty}$ is monotonic nondecreasing and converges to a finite limit as $N \to \infty$.

Lemma 2 *If retailer II uses a stationary base stock strategy with order-upto-level z_2, then*
(i) *for every $n = 1, 2, ...$, $D_{1n}(z_1, z_2)$ is minimized at a finite z_1 value,*
(ii) $S_{1(n+1)} \geq S_{1n}$ *for* $n = 1, 2, ...$,
(iii) $\{S_{1n}\}_{n=1}^{\infty}$ *is convergent.*

Proof:

(i) The proof is given by induction. In the analysis of the curve $\frac{\partial}{\partial z_1} D_{11}(z_1, z_2) = 0$ in the (z_1, z_2) plane, implicit differentiation gives the derivative of z_2 with respect to z_1, which will be denoted by $\frac{dz_2^{11}}{dz_1}$. The superscript 11 is used because it is obtained from the first retailer's cost function in period $n = 1$. $\frac{dz_2^{11}}{dz_1}$ is derived from

$$l_1 f(z_1) + (q_1 + h_1) f(z_1) G(z_2) + (q_1 + h_1)\left(\frac{dz_2^{11}}{dz_1} + \frac{1}{b}\right)\int_0^{z_2} g(z_2 + \frac{z_1 - x}{b}) f(x) dx = 0.$$

Since

$$\frac{dz_2^{11}}{dz_1} = \frac{-1}{b} - \frac{l_1 f(z_1) + (q_1 + h_1) f(z_1) G(z_2)}{(q_1 + h_1)\int_0^{z_1} g(z_2 + \frac{z_1 - x}{b}) f(x) dx},$$

is negative, the curve $\frac{\partial}{\partial z_1} D_{11}(z_1, z_2) = 0$ is strictly decreasing in the (z_1, z_2) plane. Thus, given any z_2 a lower bound for the optimal z_1 value is obtained by letting z_2 go to infinity in $\frac{\partial}{\partial z_1} D_{11}(z_1, z_2) = 0$. Denote this lower bound by $\underline{z}_{11}$. It satisfies $\int_0^{\underline{z}_{11}} f(x)dx = \frac{q_1 + l_1 - c_1}{q_1 + l_1 + h_1}$. Similarly, for any given z_2 let $\bar{z}_{11}$ denote the highest value of the optimal z_1. $\bar{z}_{11}$ is the solution of $\frac{\partial}{\partial z_1} D_{11}(z_1, 0) = 0$ where z_2 is equal to zero, i.e.,

$$(q_1 + h_1)\int_0^{\bar{z}_{11}} G(\frac{\bar{z}_{11} - x}{b}) f(x) dx + l_1 F(\bar{z}_{11}) = q_1 + l_1 - c_1.$$

Since $\frac{q_1 + l_1 - c_1}{q_1 + l_1 + h_1} < 1$, it is observed that $\underline{z}_{11} < \infty$. Also, since $\frac{dz_2^{11}}{dz_1} < 0$ at any (z_1, z_2), $\bar{z}_{11} < \infty$. Hence, given any $z_2 \in [0, \infty)$ the convex cost function $D_{11}(z_1, z_2)$ is minimized at a finite z_1 value, say S_{11}, between $\underline{z}_{11}$ and $\bar{z}_{11}$.

Assuming that (i) is true for period n, the curve $\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2) = 0$ is analyzed in the (z_1, z_2) plane in order to show that (i) also holds for period $(n+1)$. The last term of $D_{1(n+1)}(z_1, z_2)$ in (2) vanishes for $0 \le z_1 < S_{1n}$ and the first partial derivative of $D_{1(n+1)}(z_1, z_2)$ becomes $A_1(z_1, z_2)$. For that reason, before proceeding with the second step of induction, behavior of $A_1(z_1, z_2)$ is investigated. Implicit differentiation of $A_1(z_1, z_2) = 0$ gives

$$\frac{dz_2^1}{dz_1} = \frac{-1}{b} - \frac{l_1 f(z_1) + (q_1 + h_1 - \gamma c_1) f(z_1) G(z_2)}{(q_1 + h_1 - \gamma c_1) \int_0^{z_1} g(z_2 + \frac{z_1 - x}{b}) f(x) dx}$$

which has the same form as the derivative obtained for $n = 1$ except that coefficient $(q_1 + h_1 - \gamma c_1)$ is included instead of $(q_1 + h_1)$. Since $q_1 > c_1$ and $0 < \gamma < 1$, the curve $A_1(z_1, z_2) = 0$ is strictly decreasing in the (z_1, z_2) plane. Using the same reasoning for $n = 1$, it is easy to see that, given any $z_2 \in [0, \infty)$, $A_1(z_1, z_2)$ vanishes at a finite value of z_1. The lower bound $\underline{z}_1$ for the optimal value of z_1 satisfies $\int_0^{\underline{z}_1} f(x)dx = \frac{q_1 + l_1 - c_1}{q_1 + l_1 + h_1 - \gamma c_1}$ and the upper bound $\bar{z}_1$ satisfies

$$(q_1 + h_1 - \gamma c_1) \int_0^{\bar{z}_1} G(\frac{\bar{z}_1 - x}{b}) f(x) dx + l_1 \int_0^{\bar{z}_1} f(x) dx = q_1 + l_1 - c_1.$$

Note that $\underline{z}_{11} < \underline{z}_1$. For a given z_2, let $z_{1|z_2}$ be the value of z_1 at which $A_1(z_1, z_2)$ is equal to zero. From the observations above, $z_{1|z_2} < \infty$. Also, since

$$A_1(z_1, z_2) = \frac{\partial}{\partial z_1} D_{11}(z_1, z_2) - \gamma c_1 \int_0^{z_1} \int_0^{z_2 + \frac{z_1 - x}{b}} g(y) f(x) dy dx.$$

and $\frac{\partial}{\partial z_1} D_{11}(z_1, z_2)$ is zero at $z_1 = S_{11}$ and $S_{11} > 0$,

$$A_1(S_{11}, z_2) = -\gamma c_1 \int_0^{S_{11}} \int_0^{z_2 + \frac{S_{11} - x}{b}} g(y) f(x) dy dx < 0.$$

The relation above shows that $S_{11} < z_{1|z_2}$ because $D_{11}(z_1, z_2)$ is a convex function of z_1.

By induction, assume that given any $z_2 \in [0, \infty)$, $D_{1n}(z_1, z_2)$ attains its minimum at a finite value S_{1n} less than or equal to $z_{1|z_2}$, i.e., $\frac{\partial}{\partial z_1} D_{1n}(z_1, z_2) = 0$ at $z_1 = S_{1n}$ such that $S_{1n} \le z_{1|z_2}$. Note that, the integration term in (3) vanishes at $z_1 = S_{1n}$. Over the range of x and y values given by the double integration term in (3), $(z_1 - x - b[y - z_2]^+) \ge S_{1n}$ holds if $z_1 > S_{1n}$. Since $D_{1n}(z_1, z_2)$ is a convex function of z_1 and its minimum is achieved at S_{1n}, $\frac{\partial}{\partial z_1} D_{1n}(w, z_2)$ is nonnegative at $w = z_1 - x - b[y - z_2]^+$ such that $w \ge S_{1n}$. Hence, the integration term in (3) is nonnegative for $z_1 > S_{1n}$.

On the other hand, since $A_1(z_1, z_2)$ is the first partial derivative of a convex function as pointed out by remark 1 and $A_1(z_{1|z_2}, z_2) = 0$, $A_1(z_1, z_2) \ge 0$ for $z_1 \ge z_{1|z_2}$. Then, it is observed that $\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2) \ge 0$ for $z_1 \ge z_{1|z_2}$. This shows that $D_{1(n+1)}(z_1, z_2)$ is nondecreasing in z_1 for $z_1 \ge z_{1|z_2}$. By the convexity of $D_{1(n+1)}(z_1, z_2)$, the minimizing point of $D_{1(n+1)}(z_1, z_2)$, i.e., $S_{1(n+1)}$, is less than or equal to $z_{1|z_2}$.

(ii) Since $D_{1(n+1)}(z_1, z_2)$ is a convex function and its minimum is achieved at $z_1 = S_{1(n+1)}$, it is sufficient to show that $\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2) < 0$ for $0 \le z_1 < S_{1n}$ and every $n \in \{1, 2, ...\}$. When $0 \le z_1 \le S_{1n}$, the first partial derivative $\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2)$ is equal to $A_1(z_1, z_2)$. Since $A_1(z_1, z_2)$ is the derivative of a convex function and $S_{1n} \le z_{1|z_2}$ from (i), $\frac{\partial}{\partial z_1} D_{1(n+1)}(z_1, z_2) < 0$ for $0 \le z_1 < S_{1n}$. Then, by the use of convexity of $D_{1(n+1)}(z_1, z_2)$ in z_1, $S_{1(n+1)} \ge S_{1n}$ holds.

(iii) Convergence of $\{S_{1n}\}_{n=1}^{\infty}$ results from the observation that $\{S_{1n}\}_{n=1}^{\infty}$ is a monotonic nondecreasing sequence in compact space $[0, z_{1|z_2}]$. $\square$

Over a finite horizon, say N-period horizon, order-upto-levels S_{1n} for $n = 1, ..., N$, form optimal non-stationary base stock strategy of retailer I given the second retailer's stationary base stock strategy with order-upto-level z_2. In order to determine optimal strategy of the first retailer over infinite horizon, the limiting behavior of the payoff function $C_{1n}(I, z_2)$ is analyzed and the corresponding functional equation is given in lemma 3.

Lemma 3 *Given the second retailer's stationary base stock strategy with order-upto-level z_2, $C_{1n}(I, z_2)$ converges uniformly for all I in a finite interval. The limit function $C_1(I, z_2)$ is a continuous function of I and it is the unique bounded solution to*

$$C_1(I, z_2) = \min_{z_1 \geq I} \left\{ c_1(z_1 - I) + L_1(z_1, z_2) + \gamma \int_0^\infty \int_0^\infty C_1([z_1 - x - b[y - z_2]^+]^+, z_2)\, g(y) f(x) dy dx \right\}.$$

Proof: In lemma 2(i), it is shown that, for any $z_2 \in [0, \infty)$, an upper bound for $S_{1n} \geq 0$, $n = 1, 2, ...$, is $z_{1|z_2}$ which is given by the solution of $A_1(z_{1|z_2}, z_2) = 0$. Also, $\bar{z}_1 < \infty$ is an upper bound for $z_{1|z_2}$. Since the expected values $E(X)$ and $E(Y)$ are also assumed to be finite, $|C_{1n}(I, z_2)|$ is bounded for all I in $[0, \bar{z}_1]$.

In order to establish the convergence of $C_{1n}(I, z_2)$, the notation and the method introduced by Bellman, Glicksberg and Gross [1] and later used by Iglehart [3] is considered. Let T_1 be the operator defined as follows:

$$T_1(z_1, I, C_1|z_2) \quad = \quad c_1(z_1 - I) + L_1(z_1, z_2) + \gamma \int_0^\infty \int_0^\infty C_1([z_1 - x - b[y - z_2]^+]^+, z_2)\, g(y) f(x) dy dx.$$

By assuming $C_{10}(I, z_2) = 0$ for every $I \geq 0$ the optimality equation can be written as $C_{1(n+1)}(I, z_2) = \min_{z_1 \geq I} \{T_1(z_1, I, C_{1n}|z_2)\}$ for every n. Let z_{1n}^I denote the optimal z_1 value given the initial inventory is I. Note that

$$z_{1n}^I = \begin{cases} S_{1n} & if \;\; I < S_{1n}, \\ I & if \;\; I \geq S_{1n}. \end{cases}$$

By the optimality of $z_{1(n+1)}^I$ and z_{1n}^I in periods $(n+1)$ and n, respectively,

$$\begin{aligned} C_{1(n+1)}(I, z_2) &= T_1(z_{1(n+1)}^I, I, C_{1n}|z_2) \leq T_1(z_{1n}^I, I, C_{1n}|z_2), \\ C_{1n}(I, z_2) &= T_1(z_{1n}^I, I, C_{1(n-1)}|z_2) \leq T_1(z_{1(n+1)}^I, I, C_{1(n-1)}|z_2), \end{aligned}$$

hold. These relations give the following upper bound for the difference between $C_{1(n+1)}(I, z_2)$ and $C_{1n}(I, z_2)$:

$$|C_{1(n+1)}(I, z_2) - C_{1n}(I, z_2)| \leq \max_{k=n,n+1} \left\{ |T_1(z_{1k}^I, I, C_{1n}|z_2) - T_1(z_{1k}^I, I, C_{1(n-1)}|z_2)| \right\}$$

$$\leq \max_{k=n,n+1} \left\{ \begin{array}{l} \gamma \int_0^\infty \int_0^\infty |C_{1n}([z_{1k}^I - x - b[y - z_2]^+]^+, z_2) \\ \qquad - C_{1(n-1)}([z_{1k}^I - x - b[y - z_2]^+]^+, z_2)|\, g(y) f(x) dy dx \end{array} \right\}.$$

Then, the above inequality is rewritten as

$$\max_{0 \leq I \leq \bar{z}_1} \left\{ |C_{1(n+1)}(I, z_2) - C_{1n}(I, z_2)| \right\} \quad \leq \quad \gamma \max_{0 \leq I \leq \bar{z}_1} \left\{ |C_{1n}(I, z_2) - C_{1(n-1)}(I, z_2)| \right\}.$$

Using this relation successively, one obtains

$$\max_{0 \leq I \leq \bar{z}_1} \left\{ |C_{1(n+1)}(I, z_2) - C_{1n}(I, z_2)| \right\} \quad \leq \quad \gamma^n \max_{0 \leq I \leq \bar{z}_1} \left\{ |C_{11}(I, z_2)| \right\}, \quad n = 1, 2, ...$$

Since $|C_{11}(I, z_2)|$ is bounded for all I in $0 \leq I \leq \bar{z}_1$, the series $\sum_{n=0}^\infty |C_{1(n+1)}(I, z_2) - C_{1n}(I, z_2)|$ converges for $0 \leq I \leq \bar{z}_1$, which means that the series $\sum_{n=0}^\infty (C_{1(n+1)}(I, z_2) - C_{1n}(I, z_2))$ converges absolutely. This implies that $\lim_{n\to\infty}(C_{1(n+1)}(I,z_2) - C_{1n}(I,z_2)) = 0$. As a result, $C_{1n}(I,z_2)$ converges for every I in the finite interval $[0,\bar{z}_1]$. Note that the convergence is uniform.

From (1), one can easily observe that $C_{1n}(I,z_2)$ is a continuous function. This leads to the continuity of the limit function $C_1(I,z_2)$.

In order to show that $C_1(I,z_2)$ satisfies the functional equation given in the lemma, consider

$$\lim_{n\to\infty}\min_{z_1\geq I}\left\{T_1(z_1,I,C_{1(n-1)}|z_2)\right\} = \lim_{n\to\infty} C_{1n}(I,z_2).$$

Note that $z_1 \leq \bar{z}_1$. For any I, $T_1(z_1,I,C_{1(n-1)}|z_2)$ is a continuous function of z_1. Hence, the limit and the minimization operations can be interchanged as follows:

$$\min_{z_1\geq I}\left\{\lim_{n\to\infty} T_1(z_1,I,C_{1(n-1)}|z_2)\right\} = C_1(I,z_2).$$

For I being restricted to the interval $[0,\bar{z}_1]$, by the bounded convergence theorem, the limit operation and the double integral in $T_1(z_1,I,C_{1(n-1)}|z_2)$ can be interchanged and the above relation can be written as

$$C_1(I,z_2) \;=\; \min_{z_1\geq I}\left\{T_1(z_1,I,\lim_{n\to\infty} C_{1(n-1)}|z_2)\right\} \;=\; \min_{z_1\geq I}\left\{T_1(z_1,I,C_1|z_2)\right\}.$$

Since $C_{1n}(I,z_2)$ is a contraction, by the fixed point theorem for the contraction mappings $C_1(I,z_2)$ is the unique bounded solution. $\square$

The next step is to determine the optimal strategy of retailer I for infinite horizon problem. For this purpose, behavior of $\lim_{n\to\infty} D_{1n}(z_1,z_2)$ is investigated and it is observed that this limiting function is minimized at $\lim_{n\to\infty} S_{1n}$.

Lemma 4 *Over infinite horizon, if retailer II uses a stationary base stock strategy with order-upto-level z_2, then the first retailer's optimal strategy is also a stationary base stock strategy with order-upto-level $z_{1|z_2}$.*

Proof: The proof is based on the analysis of $\lim_{n\to\infty} D_{1n}(z_1,z_2)$, i.e.,

$$\lim_{n\to\infty}\left(c_1 z_1 + L_1(z_1,z_2) + \gamma\int_0^\infty\int_0^\infty C_{1(n-1)}([z_1 - x - b[y-z_2]^+]^+, z_2)\; g(y)f(x)dydx\right).$$

Denote $\lim_{n\to\infty} D_{1n}(z_1,z_2)$ by $D_1(z_1,z_2)$. Using the bounded convergence theorem on the right hand side of the above equation, one obtains

$$D_1(z_1,z_2) = c_1 z_1 + L_1(z_1,z_2) + \gamma\int_0^\infty\int_0^\infty C_1([z_1 - x - b[y-z_2]^+]^+, z_2)\; g(y)f(x)dydx.$$

For the infinite horizon problem, $D_1(z_1,z_2)$ determines the optimal strategy of the first retailer as a response to his opponent's stationary base-stock strategy with order-upto-level z_2. $D_1(z_1,z_2)$ is convex because it is the limit of a sequence of convex functions.

From lemma 2, $\{S_{1n}\}_{n=1}^{\infty}$ is a monotonic nondecreasing sequence. In order to show that $\lim_{n\to\infty} S_{1n}$ is $z_{1|z_2}$ and it is the minimizing point of $D_1(z_1,z_2)$, one needs to prove that $z_{1|z_2}$ is the least upper bound for the range of sequence $\{S_{1n}\}_{n=1}^{\infty}$.

Consider the first partial derivative of $D_{1(n+1)}(z_1,z_2)$ in (3). At $z_1 = S_{1n}$, the derivative is equal to $A_1(S_{1n},z_2)$, which is nonpositive since $S_{1n} \leq z_{1|z_2}$. For every $0 \leq z_1 < z_{1|z_2}$, $A_1(z_1,z_2)$ is negative. This shows that a point in $[0,z_{1|z_2})$ can not be the least upper bound for the range of $\{S_{1n}\}_{n=1}^{\infty}$. Thus, since the range of $\{S_{1n}\}_{n=1}^{\infty}$ is $(0,z_{1|z_2}]$, $z_{1|z_2}$ is the least upper bound.

In order to show that $z_{1|z_2}$ is the solution for the infinite horizon problem, consider the first partial derivative of the limiting cost function $D_{1(n+1)}(z_1, z_2)$. From (3),

$$\frac{\partial}{\partial z_1} D_1(z_1, z_2) = A_1(z_1, z_2) + \gamma \int_0^{[z_1 - z_{1|z_2}]^+} \int_0^{z_2 + \frac{z_1 - x - z_{1|z_2}}{b}} \frac{\partial}{\partial z_1} D_1(z_1 - x - b[y - z_2]^+, z_2)\, g(y) f(x) dy dx.$$

For $0 \le z_1 < z_{1|z_2}$, $A_1(z_1, z_2)$ takes negative values. At $z_1 = z_{1|z_2}$, the first derivative of $D_1(z_1, z_2)$ with respect to z_1 is $A_1(z_{1|z_2}, z_2)$ which is zero. Then, since $D_1(z_1, z_2)$ is convex, $z_{1|z_2}$ is the minimizing point. □

Lemma 4 implies that if one retailer restricts himself to stationary base stock strategies and if this is declared by that retailer, his opponent can also restrict himself to the stationary base stock strategies for optimizing his payoff.

The results given above are all obtained under the assumption that retailer II uses a stationary base stock strategy. If the first retailer's strategy is given as a stationary base stock strategy, then the same results follow for retailer II. Based on these observations, in the remaining of this section it is shown that there exists a Nash equilibrium which is unique within the class of stationary base stock strategies. Below, Nash equilibrium of stationary base stock strategies is defined for the two-person nonzero-sum stochastic game formulation of the infinite horizon substitutable product inventory control problem. The payoff functions of this game are D_1 and D_2, the latter of which is given by the limit of D_{2n} as n tends to infinity.

Definition 1 (S_1^*, S_2^*) *is called a Nash equilibrium relative to initial inventory levels* $[0, S_1^*] \times [0, S_2^*]$ *if* $D_1(S_1^*, S_2^*) \le D_1(z_1, S_2^*)$, *for all* $z_1 \ge 0$, *and* $D_2(S_1^*, S_2^*) \le D_2(S_1^*, z_2)$, *for all* $z_2 \ge 0$.

Nash condition implies that, if a retailer takes his Nash strategy his opponent can not improve his payoff by taking any strategy other than his Nash strategy.

Before proceeding with the main theorem for the existence and uniqueness of a Nash equilibrium of the substitutable product inventory problem within the class of stationary base stock strategies, it should be pointed out that, from now on, the case with infinite order quantities will not be considered.

Remark 2: *If one of the retailers gives an order of infinite units, then he can satisfy every customer for his product, i.e., no one switches to the other product. In such a case, the other retailer would not have any hope of having substitutable demand and so he decides to satisfy only the demand for his product. In other words, for this retailer the problem reduces to a single player problem. The former retailer goes into bankruptcy because the expected value of demand is finite for each product. Hence, if a retailer orders infinitely many units, then his cost becomes infinite (and this is the worst he could do).*

This remark leads to another way of observing the validity of lemma 2(i) when a retailer orders infinite units. Then, his cost becomes infinite regardless of his opponent's order-upto-level, i.e., $\lim_{z_1 \to \infty} D_{1n}(z_1, z_2) = \infty$ for all $z_2 \ge 0$ and $\lim_{z_2 \to \infty} D_{2n}(z_1, z_2) = \infty$ for all $z_1 \ge 0$. □

As shown before, for any order-upto-level $z_2 \in [0, \infty)$ of the second retailer, retailer I chooses his own order-upto-level in the finite interval $[0, \bar{z}_1]$. Such bounds are obtained also for the second retailer. For a given $z_1 \in [0, \infty)$, let

$$\begin{aligned} A_2(z_1, z_2) \quad &= \quad c_2 - (q_2 + l_2) \int_{z_2}^{\infty} g(y) dy + (h_2 - \gamma c_2) \int_0^{z_2} g(y) dy \\ &\quad - (q_2 + h_2 - \gamma c_2) \int_0^{z_2} \int_{z_1 + \frac{z_2 - y}{a}}^{\infty} f(x) g(y) dx dy. \end{aligned}$$

Then, the implicit differentiation of $A_2(z_1, z_2) = 0$ gives

$$l_2 g(z_2)\frac{dz_2^2}{dz_1} + (q_2+h_2)g(z_2)F(z_1)\frac{dz_2^2}{dz_1} + (q_2+h_2)\left(1+\frac{1}{a}\frac{dz_2^2}{dz_1}\right)\int_0^{z_2} f(z_1+\frac{z_2-y}{a})g(y)dy = 0.$$

From this relation, one obtains

$$\frac{dz_2^2}{dz_1} = \frac{-(q_2+h_2-\gamma c_2)\int_0^{z_2} f(z_1+\frac{z_2-y}{a})g(y)dy}{(q_2+h_2-\gamma c_2)\left(g(z_2)F(z_1)+\int_0^{z_2} f(z_1+\frac{z_2-y}{a})\frac{g(y)}{a}dy\right)+l_2 g(z_2)}.$$

By symmetry, $A_2(z_1, z_2) = 0$ is also a strictly decreasing curve in the (z_1, z_2) plane. This can be seen by observing the validity of the discussion in the proof of lemma 2(i) when z_1 is fixed in $D_{2(n+1)}(z_1, z_2)$. The lower bound $\underline{z}_2$ for $z_{2|z_1}$ is given by the solution of $\lim_{z_1\to\infty} A_2(z_1, z_2) = 0$. Then, $\int_0^{\underline{z}_2} g(y)dy = \frac{q_2+l_2-c_2}{q_2+l_2+h_2-\gamma c_2} < 1$ and so $\underline{z}_2$ is finite. Similarly, the upper bound $\bar{z}_2$ is obtained when $z_1 = 0$ in $A_2(z_1, z_2) = 0$. Also, $\bar{z}_1$ is finite because $\frac{dz_2^2}{dz_1} < 0$ and $\underline{z}_1$ is finite.

Nash strategies of the retailers within the class of stationary base stock strategies are characterized in theorem 1.

Theorem 1 *The infinite horizon substitutable product inventory control problem has a Nash equilibrium characterized by stationary order-upto-levels, say S_1^* and S_2^*, relative to the initial inventory levels $I \le S_1^*$ and $J \le S_2^*$ of retailers I and II, respectively. This is the unique Nash equilibrium within the class of stationary base stock strategies.*

Proof: Suppose that (S_1^*, S_2^*) is a solution of $A_1(z_1, z_2) = 0$ and $A_2(z_1, z_2) = 0$ for (z_1, z_2). Namely, $S_1^* = z_{1|S_2^*}$ and $S_2^* = z_{2|S_1^*}$. From lemma 4, given S_2^* as the order-upto-level of the second retailer's stationary base stock strategy, $D_1(z_1, S_2^*)$ is a convex function that is minimized at $z_{1|S_2^*}$. Recall that $z_{1|S_2^*}$ is the solution of $A_1(z_1, S_2^*) = 0$. Hence, one condition of Nash equilibrium, namely $D_1(S_1^*, S_2^*) \le D_1(z_1, S_2^*)$, for all $z_1 \ge 0$, is satisfied at $S_1^* = z_{1|S_2^*}$. Similarly, given S_1^* as the first retailer's order-upto-level, $D_2(S_1^*, z_2)$ is convex and its minimizing point $z_{2|S_1^*}$ is obtained by solving $A_2(S_1^*, z_2) = 0$. Thus, the other Nash condition also holds, i.e., $D_2(S_1^*, S_2^*) \le D_2(S_1^*, z_2)$, for all $z_2 \ge 0$, $S_2^* = z_{2|S_1^*}$.

The next step is to address the question on the existence of a pair (S_1^*, S_2^*) that satisfies both $A_1(S_1^*, S_2^*) = 0$ and $A_2(S_1^*, S_2^*) = 0$. Recall from the proof of lemma 2(i) that the curve $A_1(z_1, z_2) = 0$ is strictly decreasing, and for every z_2 in $[0, \infty)$, $z_{1|z_2}$ takes values between $\underline{z}_1$ and $\bar{z}_1 < \infty$. Similarly, given any $z_1 \in [0, \infty)$, the analysis of $A_2(z_1, z_2) = 0$ gives the lower and upper bounds $\underline{z}_2$ and $\bar{z}_2 < \infty$, respectively, for $z_{2|z_1}$. For the existence and uniqueness of the Nash solution one needs to show that there exists only one point, (S_1^*, S_2^*), at which both $A_1(S_1^*, S_2^*) = 0$ and $A_2(S_1^*, S_2^*) = 0$ hold. This is true only if the curve $A_1(z_1, z_2) = 0$ is decreasing faster than $A_2(z_1, z_2) = 0$ in the (z_1, z_2) plane. Compare $\frac{dz_2^1}{dz_1}$ and $\frac{dz_2^2}{dz_1}$ using the method in [9]. Let

$$K = (q_1+h_1-\gamma c_1)f(z_1)G(z_2) > 0,\quad L = l_1 f(z_1) > 0,\quad Z = l_2 g(z_2) > 0,$$
$$M = (q_1+h_1-\gamma c_1)\int_0^{z_1} g(z_2+\frac{z_1-x}{b})f(x)dx > 0,$$
$$R = (q_2+h_2-\gamma c_2)\int_0^{z_2} f(z_1+\frac{z_2-x}{a})g(y)dy > 0,$$
$$T = (q_2+h_2-\gamma c_2)F(z_1)g(z_2) > 0,$$

then, $\frac{dz_2^1}{dz_1}$ and $\frac{dz_2^2}{dz_1}$ are written as $\frac{dz_2^1}{dz_1} = -\frac{1}{b} - \frac{(K+L)}{M}$, $\frac{dz_2^2}{dz_1} = \frac{-R}{T+\frac{R}{a}+Z}$. The difference of the derivatives is

$$\frac{dz_2^2}{dz_1} - \frac{dz_2^1}{dz_1} = \frac{M(T+Z) + b(K+L)(T+\frac{R}{a}+Z) + (\frac{1}{a}-b)RM}{bM(T+\frac{R}{a}+Z)},$$

which is positive since every term both in the numerator and the denominator are positive. Hence, there exists a unique intersection of the curves $A_1(z_1, z_2) = 0$ and $A_2(z_1, z_2) = 0$ in (z_1, z_2) plane. $\square$

Remark 3: *Nash equilibrium identified above for the infinite horizon problem is myopic because it is the Nash solution of the static (one-period) game with the following payoff functions of the retailers for every* (z_1, z_2) *pair:*

$$c_1 z_1 + L_1(z_1, z_2) - \gamma c_1 \int_0^{z_1} \int_0^{z_2 + \frac{z_1 - x}{b}} \left(z_1 - x - b[y - z_2]^+\right) \, g(y) f(x) dy dx, \tag{4}$$

$$c_2 z_2 + L_2(z_1, z_2) - \gamma c_2 \int_0^{z_2} \int_0^{z_1 + \frac{z_2 - y}{a}} \left(z_2 - y - a[x - z_1]^+\right) \, f(x) g(y) dx dy. \tag{5}$$

The model developed in this section satisfies the conditions presented by Sobel in [13] to guarantee the existence of myopic equilibrium strategies in stochastic games with finite state and action spaces.

Below, an explanation is given for the satisfaction of each condition:
(i) The instantaneous payoff function is the summation of two terms, one term is a function of the actions taken and the other is a function of the current state as shown below:

$$\begin{aligned} -r^1_{(I,J)(z_1-I)(z_2-J)} &= \{c_1 z_1 + L_1(z_1, z_2)\} - c_1 I, \\ -r^2_{(I,J)(z_1-I)(z_2-J)} &= \{c_2 z_2 + L_2(z_1, z_2)\} - c_2 J; \end{aligned}$$

(ii) The transition probabilities do not depend on the current state but on the actions taken as shown in (**??**);
(iii) From theorem 1, the static Nash non-cooperative game in (4), (5) has an equilibrium;
(iv) Under the equilibrium strategies of the static game, all transitions occur between the states in $[0, S_1^*] \times [0, S_2^*]$. In other words, equilibrium strategies of the static game are feasible for the states in $[0, S_1^*] \times [0, S_2^*]$.
$\square$

An option for the retailers is to cooperate for the best total payoff for both products and then share this amount. The analysis of this case is simply an extension of the discussion about cooperation in [9] to the multi-period model as shown by the following remark:

Remark 4: *Cooperation always gives better total payoff than the summation of the payoff amounts of the two retailers in the strict non-cooperative case.*

Below, four different cases are considered to compute total payoff when the retailers cooperate. Lost sale cost is not incurred if demand of one product is satisfied by the other product.

Case 1: $x_n \leq z_{1n}$, $y_n \leq z_{2n}$

$payoff_c = q_1 x_n + q_2 y_n - c_1 Q_{1n} - c_2 Q_{2n} - h_1 I_{n-1} - h_2 J_{n-1}$,
$I_{n-1} = z_{1n} - x_n$, $J_{n-1} = z_{2n} - y_n$.

Case 2: $x_n \leq z_{1n}$, $y_n > z_{2n}$

$$\begin{aligned} payoff_c = {} & q_1 min\left\{z_{1n}, x_n + b(y_n - z_{2n})\right\} + q_2 z_{2n} - c_1 Q_{1n} - c_2 Q_{2n} - h_1 I_{n-1} \\ & -l_2((1-b)(y_n - z_{2n})) - l_2 max\left\{0, b(y_n - z_{2n}) - (z_{1n} - x_n)\right\}, \end{aligned}$$

$I_{n-1} = max\left\{0, (z_{1n} - x_n) - b(y_n - z_{2n})\right\}$, $J_{n-1} = 0$.

In this case, $(1-b)(y_n - z_{2n})$ is the demand for product 2 lost because customers do not accept substitution and $max\,\{0, b(y_n - z_{2n}) - (z_{1n} - x_n)\}$ is the substitutable demand for product 2 which is lost when there is not enough stock of product 1.

Case 3: $x_n > z_{1n},\, y_n \leq z_{2n}$

$$\begin{aligned} payoff_c &= q_1 z_{1n} + q_2 min\,\{z_{2n}, y_n + a(x_n - z_{1n})\} - c_1 Q_{1n} - c_2 Q_{2n} - h_2 J_{n-1} \\ &\quad - l_1((1-a)(x_n - z_{1n})) - l_1 max\,\{0, a(x_n - z_{1n}) - (z_{2n} - y_n)\}, \\ I_{n-1} &= 0,\ J_{n-1} = max\,\{0, (z_{2n} - y_n) - a(x_n - z_{1n})\}. \end{aligned}$$

Here, $(1-a)(x_n - z_{1n})$ denotes the amount that can not be substituted by product 2 and $a(x_n - z_{1n}) - (z_{2n} - y_n)$ is the substitutable amount which is lost if it is greater than zero.

Case 4: $x_n > z_{1n},\, y_n > z_{2n}$

$$\begin{aligned} payoff_c &= q_1 z_{1n} + q_2 z_{2n} - c_1 Q_{1n} - c_2 Q_{2n}, \\ I_{n-1} &= 0,\ J_{n-1} = 0. \end{aligned}$$

Now, comparison of the non-cooperative case and cooperation of the retailers immediately leads to the observation that $payoff_c \geq payoff_1 + payoff_2$ for every possible $x_n,\ y_n,\ z_{1n},\ z_{2n}$ values. Note that $payoff_c = payoff_1 + payoff_2$ in cases 1 and 4. This means that one-period expected total payoff is greater than the summation of the expected payoffs of the retailers in the non-cooperative case, i.e.,

$$r^c_{(I,J)Q_1Q_2} \quad \geq \quad r^1_{(I,J)Q_1Q_2} + r^2_{(I,J)Q_1Q_2} \qquad for\ every\ (I,J),\ (Q_1, Q_2),$$

where r^c is defined as the one-period expected payoff when there is cooperation. Also, in each of the four cases above inventory levels at the beginning of the coming period are the same as in the non-cooperative case. Then, for any given initial inventory levels and any given strategy pair, the expected total discounted payoff in the case of cooperation is better than the summation of the individual discounted payoff amounts of the retailers incurred in the non-cooperative case. $\square$

In order to find an optimal joint strategy of the retailers when they cooperate, one needs to proceed with a single-retailer multi-product model. Finite-horizon dynamic programming formulation would then have the following form:

$$\begin{aligned} C_{cn}(I,J) = \quad & \min_{(z_1,z_2)\geq(I,J)} \quad -r^c_{(I,J)(z_1-I, z_2-J)} \\ & + \gamma \int_0^\infty \int_0^\infty C_{c(n-1)}([z_1 - x - b[y - z_2]^+]^+, [z_2 - y - a[x - z_1]^+]^+)\ g(y) f(x) dy dx, \end{aligned}$$

where $C_{cn}(I,J)$ is defined as the minimum expected discounted total payoff for the remaining n periods until the end of the horizon. In order to investigate the structure of optimal ordering strategies for this formulation or when n goes to infinity, the analysis should be performed within the context of single-retailer multi-product dynamic inventory control.

4 Conclusion

In this study, infinite horizon substitutable product inventory problem is formulated as a two-person nonzero-sum discounted stochastic game and Nash ordering strategies of the retailers are investigated within the class of stationary base stock strategies. It is assumed that the set-up costs are zero. The analysis is based on minimizing the discounted payoff function of one retailer given that the other retailer is using a stationary base stock strategy. It is shown that his optimal strategy is also a stationary base stock strategy. The existence of a unique Nash equilibrium is proved within the class of stationary base stock strategies. Also, cooperation of the retailers is observed as a solution dominating the non-cooperative solution alternatives in the sense of giving better expected total discounted payoff than the summation of the payoffs of the retailers in the non-cooperative case.

The infinite horizon model presented in this article is an extension of the single-period problem considered in [9]. Parlar conjectured the existence of (s, S)-type Nash strategies for multi-period problem in the same article. The work here in this article proves the validity of this conjecture when there is no set-up cost. The analysis also shows that the stationary base stock Nash strategies of the retailers are myopic in accordance with the results obtained in [6] for a class of dynamic oligopolies and the generalization of these results in [13].

Relaxation of the constraints under which the substitutable product inventory problem is analyzed in this article underlines future research directions as itemized below:

- The ordering cost is a linear function of the quantity ordered. Analyzing the problem when the set-up costs are nonzero and investigating the validity of Parlar's conjecture on the existence of (s, S)-type Nash strategies remain as further research subjects.
- The Nash equilibrium identified in this article can be attained if both retailers restrict themselves to stationary base stock strategies. Analysis of the substitutable product inventory problem over a larger strategy space would address the question on the existence of other Nash strategies of different types.
- The discount factor, demand distributions and the substitution probabilities are considered stationary. However, there may be cases where those are non-stationary, e.g., the substitution probabilities might change over time as a function of the actions taken by the retailers. Consideration of the problem under such non-stationary conditions would also lead to the investigation of the problem over larger strategy spaces.
- A natural extension would be the analysis of the problem under the average expected payoff criterion.
- In proceeding along any further research direction, cooperation of the retailers would turn out as an implementable option to be studied as compared to the non-cooperative case.

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