RESIDUALS FOR k-STEP-AHEAD CONTROLLED PROCESSES AND RELATED MONITORING PROBLEMS

Edward Nugent[1], Melike Baykal-Gürsoy[2] and Kemal Gürsoy[3]

[1] IBM, Burlington, Vermont
[2] Industrial & Systems Engineering Department, Rutgers, the State University of New Jersey
[3] Management & Information Systems Department, Long Island University, Brooklyn, New York

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Abstract

A dynamical system controlled by k-step-ahead minimum variance controller is considered. Independent, identically distributed one-step-ahead process residuals are given for use in statistical process monitoring schemes. Problems encountered in the application of the monitoring schemes are discussed, particularly with respect to detecting process upsets. Upsets may occur in any of three ways, for which expressions are derived. It is shown that the mechanism by which upsets occur influences the ability of the residuals to detect the upsets. It is also shown that the effect of the disturbance on the residuals is independent of the process time delay k. The ability of the residuals to detect a change in the process dispersion is discussed. It is shown that the disturbance dynamics do not alter this ability. This information is useful in obtaining accurate estimates of control chart performance and directing the SPC practitioner in modifying the control chart design.

keywords: automatic control, quality control, minimum variance control, process monitoring.

1. INTRODUCTION

The industrial activities that transform resources into finished goods are dynamic processes exibiting time correlated output behavior. In this process, one may deviate from the desired design parameters due to controllable and uncontrollable causes, thus, producing products with highly variable output characteristics. Common automatic process control (APC) techniques apply *feedback control loops* to compensate for this behavior. Compensation takes the form of adjustments in one or more manipulable variables in order to reduce the variability of one or more output characteristics. In effect, the variability is transferred to the manipulable variables.

Sometimes the variability is due to process upsets (system variations due to special causes) which deviate from the underlying process models. The APC approach described above is short-term in that it does nothing to remove the causes of the unwanted variation. In such cases it may be more effective (and less costly in the long term) to remove the causes of these upsets rather than compensate for them. *Statistical process control* (SPC) techniques have been developed to aid in the detection of these upsets and in the identification of their causes. Once a process upset is detected, the process is stopped until its cause is identified and removed. But, some process upsets could not be removed, so they should be compensated via APC. Or some upsets may be of short duration, thus, making compensation a less costly alternative to SPC (see Box and Kramer [1] and Nugent [2]).

The activity of detection is also called *process monitoring*. Control charts are relatively simple tools which have been developed to facilitate process monitoring. They include Shewhart charts, exponentially weighted moving average (EWMA) charts, and cumulative sum (CUSUM) charts. The assumption usually used to evaluate the properties of these monitoring tools is that the observations are independent identically distributed (i.i.d.) random variables.

Many authors have studied the performance characteristics of these control charts. The most frequently used performance measure is the *average run length* (ARL). Vance [3] proposed a methodology for calculating the exact ARLs for CUSUM charts. Crowder [4] used the Fredholm integral approach to evaluate the ARL of the EWMA chart. Lucas and Saccucci [5], applying a Markov chain approach, showed that the ARL performance of the EWMA chart is comparable to that of the CUSUM chart, except in worst-case scenarios, in which the EWMA chart can be slower to react to process shifts. Champ and Woodall [6] also used a Markov chain technique to demonstrate that supplementary runs rules cause the Shewhart chart to be more sensitive to small shifts in the mean, although it may not be as sensitive as the CUSUM chart. The decision to use any of these standard charts is often based upon criteria other than ARL performance, like ease-of-use and prior practice.

If the data are not independent, but autocorrelated, the standard charting techniques may be ineffective for monitoring and improving process quality. Among others, Johnson and Bagshaw [7], Alwan [8] have shown that the primary consequence of positive autocorrelation is to increase the frequency of false alarms. This is equivalent to decreasing the "in-control" average run length of the control chart. Stoumbos and Reynolds [9] show that although autocorrelation can have a significant effect on the in-control performances of some EWMA charts, their relative out-of-control performances under independence are generally maintained for low to moderate levels of autocorrelation.

To deal with correlated data, Vasilopoulos and Stamboulis [10] suggested modifying the control limits to improve the performance of standard Shewhart chart. Another method for dealing with correlated data is the use of residuals chart. Alwan and Roberts [11], Montgomery and Friedman [12] and Montgomery [13] recommended fitting an appropriate time series model to the process data, deriving a sequence of residuals, and applying control chart techniques to the resulting sequence. Process upsets should appear in the residuals.

One criticism of this approach is that when positive correlation is present it may be very difficult to detect process upsets although the probability of detecting shifts almost immediately is high. Harris and Ross [14], Ryan [15], Wardell, Moskowitz, and Plante [16] and Yashchin [17], have shown that positive correlation increases the "out-of-control" ARL of the control chart. Runger, Willemain and Prabhu [18] suggested modifying standard CUSUM chart design guidelines applied to the residuals to improve the chart performance.

This approach also suffers from the necessity of finding an appropriate time series model of the process data, which may be quite difficult to obtain in practice. Montgomery and Mastrangelo [19] suggested monitoring the residuals from an EWMA which approximates the exact time series model. Note that for some models the EWMA gives a good approximation and a poor one for other models.

Recently, Runger and Willemain [20] suggested using an unweighted batch means (UBM) chart in environments with large volumes of autocorrelated observations. In this chart, data are divided into mutually exclusive batches of finite number of successive observations. Then, the chart plots the arithmetic average of each batch with the expectation that the plotted values show reduced correlation. The authors show that for an AR(1) output process, the performance of the residuals chart is degraded as the positive correlation between observations increases. The UBM chart is shown to perform better than the residuals chart in such cases. On the other hand, in some other cases UBM charts may not be advantageous(see [2]). Although, UBM chart requires no time series modeling, one should find a batch size that will give approximately uncorrelated batch averages.

Many authors have written about the integration of APC and SPC methodologies; MacGregor [21], Box and Kramer [1], Box, Jenkins, and Reinsel [22], Vander Wiel, Tucker, Faltin, and Doganaksoy [23], Vander Wiel [24], Box and Luceno [25], Capilla, Ferrer, Romero and Hualda [26], Castillo [27], and Castillo [28]. Most researchers have considered the one-step-ahead minimum variance (MV) control problem, producing output observations which are i.i.d. Although the controller may be capable of eliminating *special cause* mean shifts as well as common cause variation, this often requires increasingly large control actions. In such cases, it is desirable to detect and remove the cause of the shift. Most researchers have considered one specific way of mean shift occurance that corresponds to one of the cases investigated in this paper. In this case, the effect of mean shift on the output observations will deminish in the long run following its occurance. Some authors suggested monitoring the control variable instead of output observations to improve the chance of detection (Faltin and Tucker, Box and Kramer [29], Messina, Montgomery, Keats, and Runger [30], Tsung, Shi, and Wu [31], Jiang, Tsui, and Woodall [32], Jiang and Tsui [33]).

However, in many systems, particularly those with short sampling intervals, it is common to encounter time delays. Time delays can arise from delays in the process itself or from delays in the processing of sensed signals. For example, time delays often occur in chemical plant processes due to the time required for material to flow through pipes.

In this paper (see also an earlier version Nugent and Baykal-Gursoy [34]) we consider a dynamic process with k periods of delay. In section 2 we begin developing the system model and the k-step-ahead MV controller. In the remainder of the paper, we assume that the controller is implemented and tuned appropriately to compensate for *common cause* variation. The residuals will be employed to aid in shift detection.

In section 3, we derive general expressions for the output and residuals when process upsets enter the system in each of three ways. These may be used to model any type of disturbance process as well as any type of process upsets. With this information, the practitioner can better predict how a given monitoring scheme will perform. If the expected detection time of the proposed scheme is too long, the expressions may point the way to more sensitive methods.

2. SYSTEM MODELING

In considering the k-step-ahead control problem, we formulate the optimal control problem. This requires the specification of the process dynamics, the environment, the performance criterion, and the restrictions on the control law.

2.1 Process Model

Consider a discrete, time-invariant, linear dynamical process with one input, u, one output, y, and disturbance d. The input-output relation together with the effect of the environment may therefore be described by the following equation,

$$y_t = \frac{B'(z^{-1})}{A'(z^{-1})} u_{t-k} + d_t,$$

where $A'(z^{-1}) = 1 + a'_1 z^{-1} + \ldots + a'_m z^{-m}$, and $B'(z^{-1}) = b'_0 + b'_1 z^{-1} + \ldots + b'_m z^{-m}$. Here, the backward time shift operator, z^{-1}, is defined by $z^{-k} y_t = y_{t-k}$. In Box *et al.*'s [22, 35] notation, the backward shift operator is denoted as "B". Note also that a time delay of k sampling intervals is present in the system. There is no loss of generality in assuming polynomials are of order m since trailing coefficients may always be set equal to zero.

Assume that the disturbance d_t is a stationary Gaussian process (Brockwell and Davis [36]) that is represented as,

$$d_t = \lambda \frac{C'(z^{-1})}{A^*(z^{-1})} \epsilon_t,$$

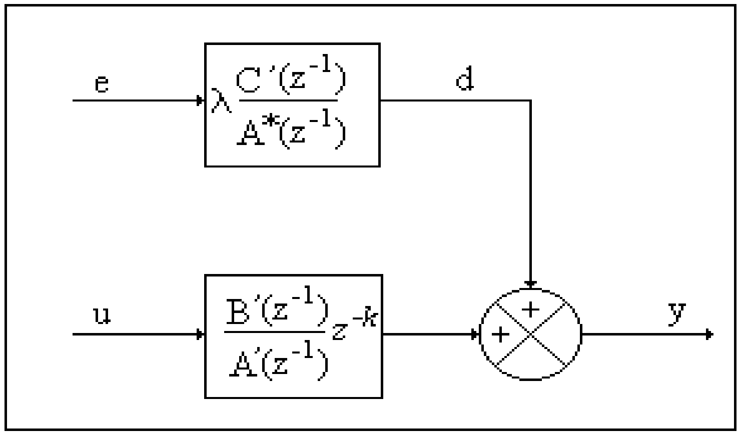

Figure 1: Model of single-input, single-output dynamical system.

where $\{\epsilon_t\}$ is a sequence of i.i.d. normal $(0, 1)$ random variables and λ is a scaling factor which specifies the dispersion of the underlying white noise process. The polynomials are $C'(z^{-1}) = 1 + c'_1 z^{-1} + \ldots + c'_m z^{-m}$, and $A^*(z^{-1}) = 1 + a^*_1 z^{-1} + \ldots + a^*_m z^{-m}$. (Stationarity of $d(t)$ requires that all zeros of the polynomial $A^*(z^{-1})$ must lie inside the unit circle, i.e., $|z_0| < 1$.)

Therefore, the entire system is represented by the equation as illustrated graphically in Figure 1,

$$y_t = \frac{B'(z^{-1})}{A'(z^{-1})} u_{t-k} + \lambda \frac{C'(z^{-1})}{A^*(z^{-1})} \epsilon_t. \tag{1}$$

By defining the following polynomials,

$$A(z^{-1}) = 1 + a_1 z^{-1} + \ldots + a_n z^{-n} = A'(z^{-1}) A^*(z^{-1}),$$

$$B(z^{-1}) = b_0 + b_1 z^{-1} + \ldots + b_n z^{-n} = B'(z^{-1}) A^*(z^{-1}),$$

$$C(z^{-1}) = 1 + c_1 z^{-1} + \ldots + c_n z^{-n} = C'(z^{-1}) A'(z^{-1}),$$

the model may be rewritten in the simplified form as,

$$A(z^{-1}) y_t = B(z^{-1}) u_{t-k} + \lambda C(z^{-1}) \epsilon_t, \tag{2}$$

thus making the output process an ARMAX (Auto Regressive Moving Average with eXogenous input) process. Again note that there is no loss of generality in assuming that all polynomials are order n, since trailing coefficients may always be set equal to zero.

2.2 Minimum Variance Control

Although the performance criterion for determining the optimal controller can be chosen in any number of ways, the typical approach is to minimize a quadratic cost function. One of the most frequently used quadratic cost functions is the variance of the output y. The optimal control law that minimizes the output variance is the minimum variance (MV) controller. It should have the property that the value of u at time t is a function only of the outputs and the control signals up to and including time t.

The solution to the MV control problem is well-defined and is given by,

$$u_t = -\frac{G(z^{-1})}{B(z^{-1})F(z^{-1})} y_t, \tag{3}$$

where the polynomials F and G of orders $k-1$ and $n-1$ respectively are defined by the following Diophantine-Aryabhatta-Bezout identity,

$$C(z^{-1}) = A(z^{-1})F(z^{-1}) + z^{-k}G(z^{-1}). \tag{4}$$

Clearly, $F(z^{-1})$ is the quotient when $C(z^{-1})$ is divided by $A(z^{-1})$, and $z^{-k}G(z^{-1})/A(z^{-1})$ is the remainder. These polynomials can be obtained using long division.

Substituting Eq. 3 and Eq. 4 into Eq. 2 results in the regulation error of the MV system,

$$\begin{aligned} y_t &= \lambda F(z^{-1})\epsilon_t \\ &= \lambda(1 + f_1 z^{-1} + \ldots + f_{k-1} z^{-(k-1)})\epsilon_t. \end{aligned} \tag{5}$$

Under the MV control law, the controlled output is a moving average of order $k-1$. Thus, if the time delay is one unit, the controlled output exhibits i.i.d. behavior, since $y_t = \lambda\epsilon_t$. Although, MV controller is valid even if it is unstable, the output variance increases for $|f_i| > 1$, since $Var[y] = \lambda^2(1 + f_1^2 + f_2^2 + \ldots + f_{k-1}^2)$. Here, stability of a controller means that $u(t)$ cannot be represented as an ever decreasing sequence of past outputs. Note that, for $k = 2$, $|f_1| > 1$ corresponds to a zero of $F(z^{-1})$ that is outside the unit circle (an unstable zero), thus, giving an unstable controller.

2.3 Residuals For k-Step-Ahead Minimum Variance Controllers

Most researchers assume a system in which control actions affect the process output in the period immediately following the control action. This is equivalent to assuming that the process attains a steady-state during the interval between sampling instances. It is frequently encountered in parts manufacturing applications and, if the sampling interval is long relative to the effects of the process dynamics, the process industries as well. The output of such a system under MV control is a sequence of i.i.d. white noise deviates.

However, we are interested in the general case when there exists one or more periods of time delay in the system. Eq. 5 indicates that the output of a system with k periods of delay is a moving average of order $k-1$. Thus, the output observations are autocorrelated and will likely produce an increase in the number of false alarms if incorporated into the process monitoring scheme.

The residuals, r_t, are defined as the difference between the actual output, y_t, and the one-step-ahead forecasts of the output at time t, made at time $t-1$, $\hat{y}_{t|t-1}$. The MV one-step-ahead predictor of the output is described by,

$$\begin{aligned}\hat{y}_{t|t-1} &= E[y_t|\epsilon_{t-1}, \epsilon_{t-2}, \ldots] = \lambda(f_1\epsilon_{t-1} + f_2\epsilon_{t-2} + \ldots + f_{k-1}\epsilon_{t-k+1}) \\ &= \lambda(F-1)\epsilon_t, \end{aligned} \tag{6}$$

where we have compacted our notation by omitting the backward shift operator, z^{-1}. In what follows, all polynomials remain functions of z^{-1} unless otherwise noted. Combining Eq. 5 and Eq. 6, we obtain the system residuals, r_t, given by,

$$\begin{aligned} r_t &= y_t - \hat{y}_{t|t-1} \\ &= \lambda F\epsilon_t - \lambda(F-1)\epsilon_t \\ &= \lambda\epsilon_t. \end{aligned} \tag{7}$$

These residuals are i.i.d. and normally distributed about zero mean with constant variance λ^2. Using the residuals, we can apply the SPC techniques mentioned above without violating any of the assumptions about the nature of the observations. We also see that the residuals are independent of the process time delay, k. Note that if the model parameters cannot be known perfectly the residuals will not be white, since the MV controller depends on these estimates. For a fuller discussion of the MV controller, the reader is directed to a stochastic control text, e.g., Åström [37], and [35, 22].

3. PROCESS MONITORING USING RESIDUALS

Control charts designed using the residuals described in Eq. 7 will demonstrate "in-control" ARLs exactly like those of standard control charts. However, the "out-of-control" ARLs will be altered by the system dynamics. In [14-16, 18, 20], this is shown for the case when the output observations are modeled by among others, an AR(1) process. In that case, large positive autocorrelation resulted in very large ARLs for the residuals chart. Runger and Willemain [20] recommended using the UBM chart to overcome this deficiency. Unfortunately, [20] only considered the AR(1) process. Other output processes may provide very different results. For example, the authors are aware of at least one case of the ARMA(1,1) model in which the UBM charting procedure is not advantageous (see [2]). Additionally, under the UBM procedure, new batch sizes must be derived for each new process being considered.

In the discussion which follows, we examine a process subject to an additional mean shift in the disturbances. We derive expressions for the output when the shift enters in each of three ways. We also derive expressions for the residuals and demonstrate how standard SPC charts perform using these statistics.

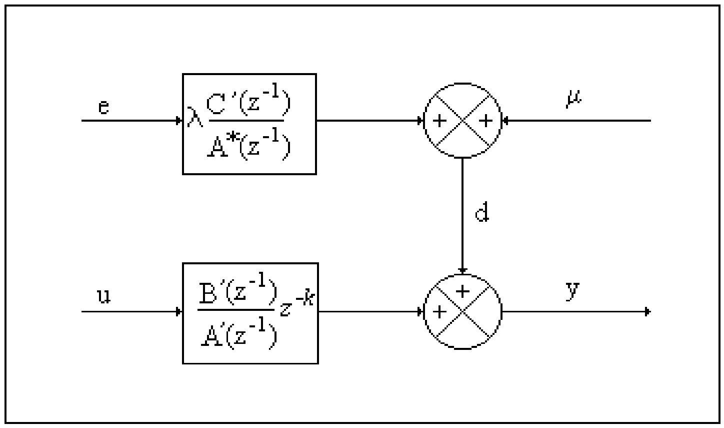

Figure 2: Model of shift entering dynamic system independently of the disturbance process

3.1 Shift Enters Independently of Disturbance Process

This case is illustrated mathematically by the following expression,

$$d_t = \lambda \frac{C'}{A^*} \epsilon_t + \mu_t, \tag{8}$$

where μ_t is the value of the mean shift at time t. Assuming a system as in Eq. 1, Figure 2 provides a graphical representation of the situation. This type of shift may be represented as a measurement bias; when detected the measurement device is recalibrated.

The presence of the process shift produces the following result,

$$\begin{aligned} y_t &= \frac{B'}{A'} u_{t-k} + \lambda \frac{C'}{A^*} \epsilon_t + \mu_t \\ &= \frac{B}{A} u_{t-k} + \lambda \frac{C}{A} \epsilon_t + \mu_t. \end{aligned} \tag{9}$$

If we assume that the system is controlled with a MV controller and apply Eq. 3 and the identity Eq. 4, we obtain the equation for the system output at time t,

$$y_t = \lambda F \epsilon_t + M_1 \mu_t, \tag{10}$$

where,

$$\begin{aligned} M_1 &= (\frac{A^* F}{C'}) \\ &= (1 + m_1^1 z^{-1} + m_2^1 z^{-2} + \ldots), \end{aligned} \tag{11}$$

and m_i^1 is the ith coefficient of the polynomial M_1. We see that the output behavior is clearly affected by the presence of the process shift. However, the magnitude of this effect depends upon the coefficients of M_1.

The one-step-ahead predictor is given by Eq. 6, which depends upon the noise innovations, ϵ_t. To implement the predictor, we make use of Eq. 7, giving,

$$\hat{y}_{t|t-1} = (F-1)r_t. \tag{12}$$

The effect of a mean shift on the process residuals may be found by combining Eq. 10 and Eq. 12,

$$\begin{aligned} r_t &= y_t - \hat{y}_{t|t-1} \\ &= \lambda F \epsilon_t + M_1 \mu_t - (F-1)r_t \\ &= \lambda \epsilon_t + \frac{A^*}{C'}\mu_t. \end{aligned} \tag{13}$$

The ability of the residuals to display the effect of the mean shift depends on the inverse dynamics of the disturbance process. It may be magnified, remain unchanged, or be reduced. We note that this ability is also independent of the process time delay, k.

Example 1

Consider a simple example with a first-order plant and a first-order autoregressive AR(1) disturbance. The temperature response of a continuously stirred tank reactor to a temperature change in the cooling water flow to its jacket is a first-order plant described by the following model,

$$y_t = \frac{\omega z^{-k}}{1-\delta z^{-1}} u_t + d_t, \tag{14}$$

where $\omega = 1-\delta$, $\delta = e^{-T/\tau}$, τ is the system time constant (time when response is e^{-1} times the initial value), and T is the discrete sampling interval. The disturbances are described by a first-order autoregressive model, defined as,

$$d_t = \frac{1}{1-\phi z^{-1}}\epsilon_t. \tag{15}$$

When $-1 < \phi < 1$, the disturbance process is said to be stationary, meaning that its joint probability distribution is unaffected by a change of time origin. Large positive values of ϕ produce series which exhibit definite trends. Large negative values of ϕ produce series which tend to oscillate rapidly. It is reasonable to suggest that disturbances to the system described above will tend to display trending behavior, and therefore, may be described by Eq. 15 where $0 < \phi < 1$.

Given a system model as in Eq. 1, the following polynomials may be derived from the definitions given by Eq. 14 and Eq. 15,

$$A(z^{-1}) = A'(z^{-1})A^*(z^{-1}) = (1-\delta z^{-1})(1-\phi z^{-1}) = (1-(\delta+\phi)z^{-1} + \delta\phi z^{-2}),$$

$$B(z^{-1}) = B'(z^{-1})A^*(z^{-1}) = \omega(1-\phi z^{-1}),$$

$$C(z^{-1}) = C'(z^{-1})A'(z^{-1}) = (1-\delta z^{-1}).$$

Eq. 4 gives $F(z^{-1}) = 1 + \phi z^{-1}$ and $G(z^{-1}) = \phi^2(1 - \delta z^{-1})$.

Let $\lambda = 1$, $k = 2$, $\phi = 0.9$, with a sampling time interval $T = -\ln(0.8)\tau = 0.223\tau$ so that $\delta = 0.8$ and $\omega = 0.2$. For such a system, applying Eq. 3 leads to the MV 2-step-ahead controller,

$$\begin{aligned} u_t &= -\frac{\phi^2(1-\delta z^{-1})}{\omega(1-\phi^2 z^{-2})} y_t \\ &= -\frac{4.05(1-0.8z^{-1})}{(1-0.81z^{-2})} y_t. \end{aligned} \tag{16}$$

If we assume that mean shifts, μ_t, enter the system from time to time as in Eq. 8, the system output becomes,

$$\begin{aligned} y_t &= \lambda F \epsilon_t + M_1 \mu_t \\ &= (1 + \phi z^{-1})\epsilon_t + (1 - \phi^{-2} z^{-2})\mu_t \\ &= (1 + 0.9z^{-1})\epsilon_t + (1 - 0.81z^{-2})\mu_t. \end{aligned} \tag{17}$$

The one-step-ahead forecasts are found from Eq. 12 as,

$$\hat{y}_{t|t-1} = (F-1)r_t = \phi r_{t-1} = 0.9 r_{t-1}, \tag{18}$$

and the process residuals are given by Eq. 13 as,

$$\begin{aligned} r_t &= \lambda \epsilon_t + \frac{A^*}{C'}\mu_t \\ &= \epsilon_t + (1 - \phi z^{-1})\mu_t \\ &= \epsilon_t + \mu_t - 0.9\mu_{t-1}. \end{aligned} \tag{19}$$

Therefore, for such a system, the "out-of-control" ARLs for an SPC monitoring scheme using the system residuals will be longer than "out-of-control" ARLs predicted using i.i.d. data. Note that this special case, Eq. 19 is similar to what was considered in [14-16, 18, 20] in the context of SPC monitoring of AR(1) process.

Figure 3 demonstrates this effect for a step and ramp disturbance shift, μ_t. These figures demonstrate the effect of the process dynamics on the ability of the residuals to detect a mean shift. In each, the size of the mean shift (in terms of σ) in the system is denoted by μ_t and the size displayed in the residuals by r_t. Note that the full effect of the step-shift is displayed in the initial residual, but is significanly reduced in later periods.

To show this in practice, suppose a 1σ step-shift enters the system at time $t = 500$. For such a system, Figure 4 shows a simulation of the output without control, the output with 2-step-ahead MV control, and the control signals for the 2-step-ahead MV controller. The controlled process demonstrates a clear improvement over the uncontrolled process. The controller is capable of compensating for both the process dynamics and the mean shift. Note, however, that a considerable

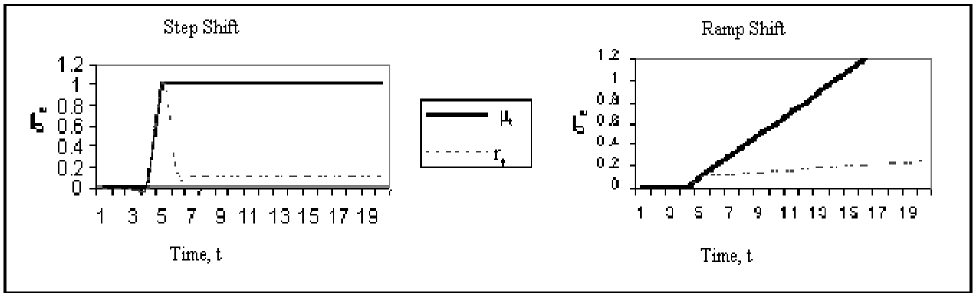

Figure 3: The effect of the process dynamics on the ability of the residuals to detect a mean shift.

amount of control effort is expended. If costs are associated with the control actions, this undesirable property could be remedied by implementation of a *suboptimal* control strategy [37, 25].

Casual observation of the controlled output and the control variable in Figure 4, by drawing the 3σ lines, shows that there will be a large number of false alarms in this system. Jiang, Tsui, and Woodall [32] introduced signal-to-noise (SN) ratios to predict the performance of SPC monitoring of one-step-ahead APC controlled process. In [33], SN ratios are used to predict the SPC chart performance of one-step-ahead Minimum Mean Squared Error (MMSE)- and Proportional and Integral- (PI) Controlled processes. Although, comparison of the SN ratios of the residuals, the controlled output and the control variable for our example suggests monitoring the output, the large number of false alarms diminish their use. This is due to the fact that for k-step-ahead MV controlled processes, the output is a moving average of order (k-1), not i.i.d. as in the case of one-step-ahead MV controlled processes.

Figure 5 shows the one-step-ahead residuals in time, an EWMA plot of the one-step-ahead residuals, and a tabular CUSUM plot, showing both the upper and lower CUSUMs of the one-step-ahead residuals (Wadsworth *et al.* [38]). Plotting the standard Shewhart chart using the one-step-ahead residuals shows that the mean shift is detected at $t = 939$, after 439 periods.

For illustrative purposes, we want the “in-control” ARL to be close to that of the Shewhart chart. One way to do this is to design the EWMA chart with the smoothing coefficient $\rho = 0.133$ and the control chart width $L = 2.777$, giving an “in-control” ARL of approximately 370. With the EWMA chart of the process residuals designed in this manner, a signal occurs in period 688, 188 periods after the step disturbance appears. Two more signals occur after the first signal in periods 801 and 939. No false alarm occurs.

The CUSUM chart also detects the shift in period 688, 188 periods after the shift appears. Again, for illustrative purposes, we want to match the “in-control” ARL to that of the EWMA chart. The decision interval is chosen to be $h = 4.1$ and the reference value $k = 0.5$. One more signal occurs at period 801. No false alarm occurs.

In all cases, we see that the time until detection is much longer than we would predict if we

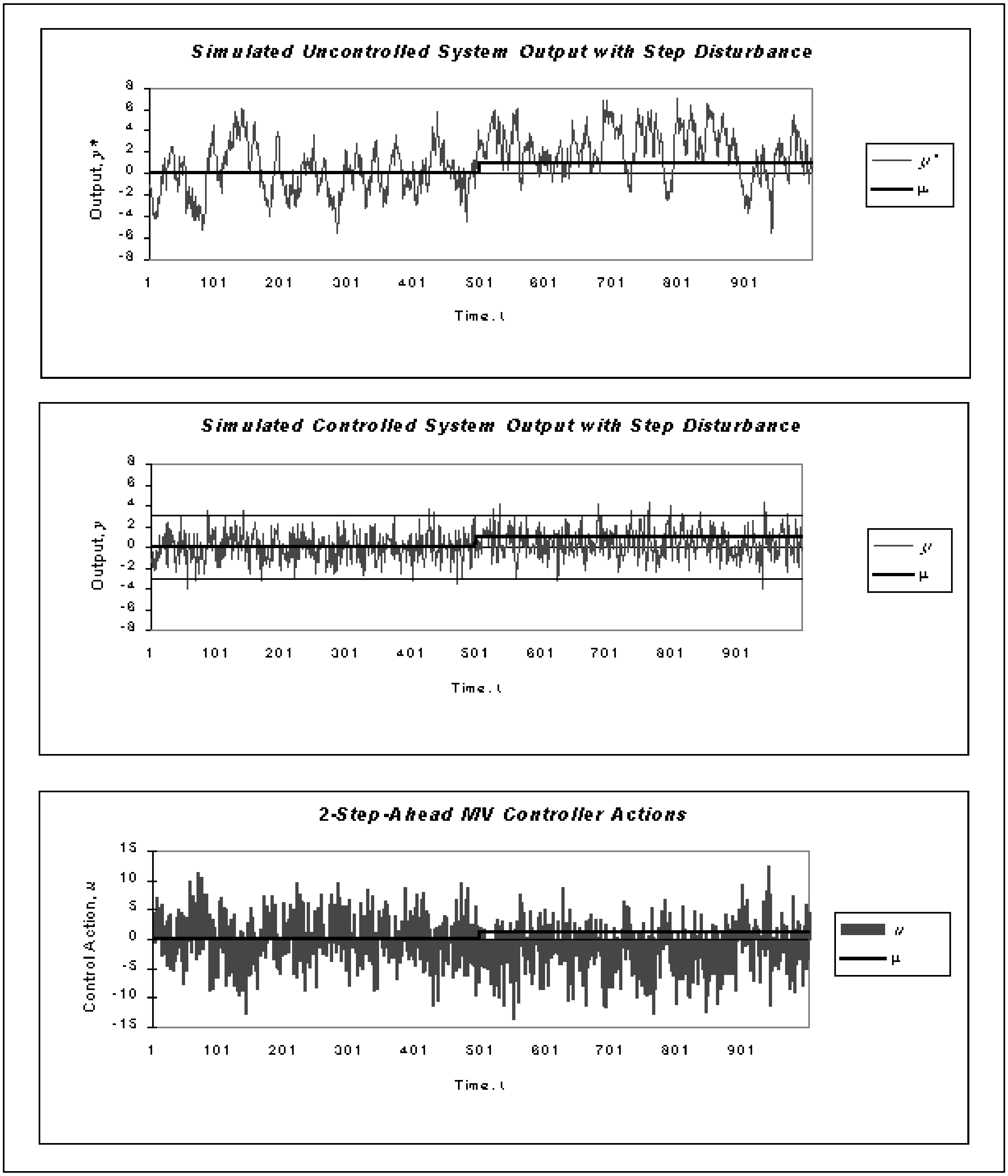

Figure 4: Simulation of the uncontrolled output, controlled output, and control signal of the system in Example 1 with $k = 2$ and the MV control strategy. A comparison between the uncontrolled and controlled outputs shows that the control strategy succeeds in reducing the output variation due to both the underlying disturbances and the additional step-shift.

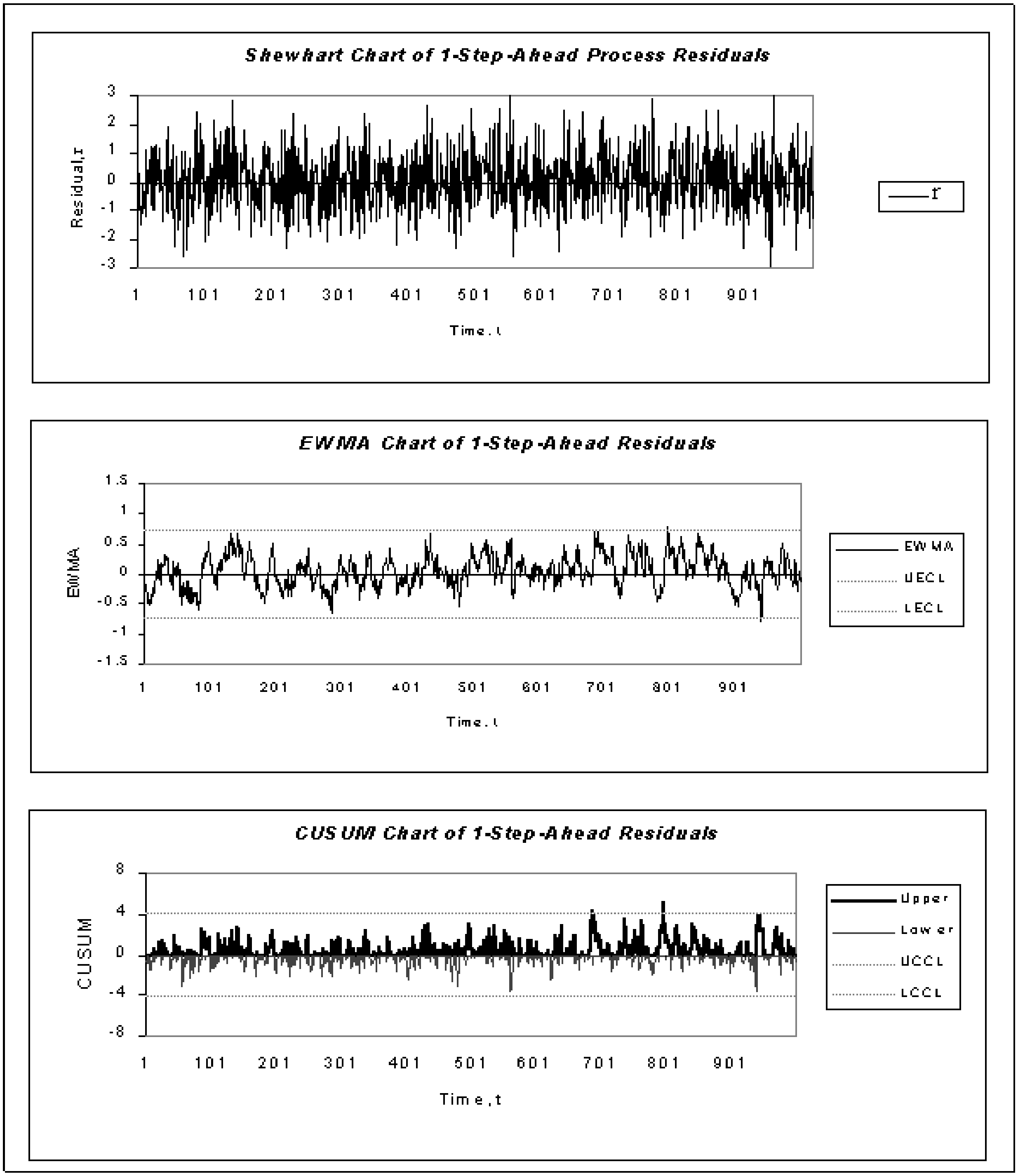

Figure 5: Shewhart, EWMA, and CUSUM charts of the residuals from the simulation results of Example 1. Casual study reveals that the standard Shewhart chart detects a change in the mean within 439 periods, the EWMA chart signals within 188 periods and the CUSUM within 188 periods. A step-shift enters the system in period 500, persisting into the observation horizon.

Method	ARL_0	Predicted $ARL_{1\sigma}$	Simulation $ARL_{1\sigma}$	Predicted $ARL_{2\sigma}$	Simulation $ARL_{2\sigma}$
Shewhart	378.9 (5.1)	43.89	318.0 (6.3)	6.3	222.4 (4.7)
EWMA	382.9 (5.8)	9.73	230.3 (4.6)	4.0	124.5 (2.5)
CUSUM	386.0 (6.2)	9.85	279.2 (5.9)	3.6	134.7 (2.6)

Table 1: Average Run Lengths for Example 1.

assume that the disturbance dynamics have no effect on the ability of the residuals to reveal a mean shift. To generalize these results, a simulation study of this system was performed in order to determine the ability of each SPC chart for residuals to detect two sizes of step disturbances. The results are based upon 2500 experimental runs. Each run is a record of observations taken until each control charting method provides a signal. Charts were implemented according to the parameter values mentioned above.

The simulation results are shown in Table 1. The second column indicates the "in-control" ARLs for each charting technique. The results in this column are based upon 2500 experimental runs, each run being a simulation of the system during which no shift occurs. They show that the charts have been properly designed to approximately match the "in-control" ARLs. The third column indicates the predicted ARL if an uncontrolled process is monitored where the outputs are i.i.d. and a $1\sigma_\epsilon$ step shift enters the system. These predicted ARLs are calculated using the methods mentioned earlier. The fourth column shows the simulated ARL until detection of a $1\sigma_\epsilon$ shift and the associated standard error. The fifth column lists the predicted ARL for charts monitoring the uncontrolled process with i.i.d. outputs when a $2\sigma_\epsilon$ step shift enters the system. In the last column, each block reports the ARL until detection of a $2\sigma_\epsilon$ step shift and the associated standard error.

The results obtained from simulation indicate that control charts designed under standard assumptions perform poorly for this system. The actual ARLs are much longer than the predicted ARLs for this shift, although both the EWMA and CUSUM charts display superior performance to the Shewhart chart.

Using the shift modification in Eq. 19 and methods for approximating ARL values, such as those [3] and [5] we may adjust our run-length predictions. In particular, predicted $1\sigma_\epsilon$ shift ARLs for each of the charts for the system of Example 1 may be updated using the following formula,

$$ARL = P_1(1) + (1 - P_1)(1 + ARL_{0.1\sigma_\epsilon}) = 1 + (1 - P_1)ARL_{0.1\sigma_\epsilon} \tag{20}$$

where $P_1 = \Pr\{$ detecting $1\sigma_\epsilon$ shift in first period$\}$ and ARL_x indicates the average run length

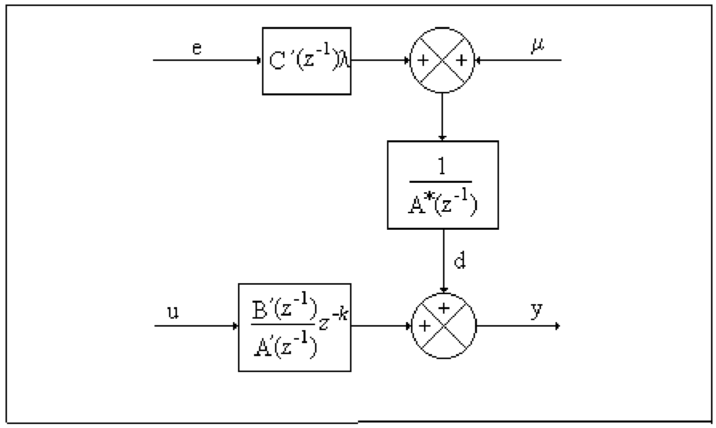

Figure 6: Model of shift entering the dynamical system within the disturbance process

given a shift of size x and we have made use of Eq. 19 to determine that a $1\sigma_\epsilon$ step-shift will degrade to $0.1\sigma_\epsilon$ in the residuals after the initial entry period into the system. In the case of the residual Shewhart chart, Eq. 20 reduces to a value of 318.6. For the residual EWMA chart, this reduces to 228.9. These values are comparable to those reported in [14, 16, 18, 20] and agree well with our simulation results. For example, in residual Shewhart charts, using the formula given in Runger and Willemain [20], we obtained an ARL value of 337.96 for $1\sigma_\epsilon$ step-shift, note that the table in their paper lists this as 345.87 which is just the ARL value of a $0.1\sigma_\epsilon$ step-shift for standard Shewhart chart. Wardell *et al.* [16] reports simulation result of 311.69 for EWMA chart when $\phi = 0.95$, via interpolation we found the ARL value for $1\sigma_\epsilon$ step-shift to be equal to 283.04.

Therefore, in order to achieve effective monitoring and shift detection, the SPC practitioner must either design control charts which are more sensitive to smaller shifts or consider other methods such as batch means [20].

3.2 Shift Occurs Within the Disturbance Process

This case is illustrated mathematically by the following expression,

$$d_t = \frac{\lambda C' \epsilon_t + \mu_t}{A^*}, \tag{21}$$

where μ_t is again the value of the mean shift at time t. Figure 6 is a graphical representation of such a situation when the system is assumed to be governed as in Eq. 1.

The presence of the shift results in the following mathematical representation of the system,

$$\begin{aligned} y_t &= \frac{B'}{A'} u_{t-k} + \lambda \frac{C'}{A^*} \epsilon_t + \frac{1}{A^*} \mu_t \\ &= \frac{B}{A} u_{t-k} + \lambda \frac{C}{A} \epsilon_t + \frac{A'}{A} \mu_t. \end{aligned} \tag{22}$$

If we again assume that the system is controlled with a MV controller and apply Eq. 3, and the identity Eq. 4, we obtain an expression for the system output at time t,

$$y_t = \lambda F \epsilon_t + M_2 \mu_t, \tag{23}$$

where,

$$M_2 = (\frac{F}{C'}) = (1 + m_1^2 z^{-1} + m_2^2 z^{-2} + \ldots).$$

The system output is again clearly affected by the presence of the disturbance shift, but its effect is dependent upon the coefficients of the polynomial M_2, and the form of the shift.

Combining Eq. 23 and the one-step-ahead predictor implementation, Eq. 12, we derive the effect of the mean shift on the process residuals,

$$r_t = y_t - \hat{y}_{t|t-1} = \lambda \epsilon_t + (C')^{-1} \mu_t. \tag{24}$$

The ability of the residuals to display the effect of the mean shift is dependent upon the inverse of the disturbance process. It may be magnified, remain unchanged, or be reduced. This ability is also independent of the process time delay, k.

Example 2

Consider the example of Section 4.1 with a mean shift occurring within the disturbance process as described above. The MV two-step-ahead controller of Eq. 16 remains unchanged and is still valid for such a system. The system output becomes,

$$\begin{aligned} y_t &= \lambda F \epsilon_t + M_2 \mu_t = (1 + \phi z^{-1}) \epsilon_t + (1 + \phi z^{-1}) \mu_t \\ &= (1 + 0.9 z^{-1}) \epsilon_t + (1 + 0.9 z^{-1}) \mu_t. \end{aligned} \tag{25}$$

The output forecasts are given by Eq. 18 and the residuals are found by application of Eq. 24,

$$r_t = \lambda \epsilon_t + (C')^{-1} \mu_t = \epsilon_t + \mu_t. \tag{26}$$

Therefore, for such a system, the "out-of-control" ARLs for an SPC scheme using the system residuals will be identical to those predicted using standard assumptions. Note that this is true for this case because $C' = 1$. Figure 7 shows the same simulation of Section 4.1 with shifts entering as part of the disturbance process. We again see the uncontrolled process output, the controlled output, and the amount of control applied. The disturbance deviates are exactly those of Example 1, including the size and duration of the ramping shift. Note that because of the disturbance dynamics, the effect of the shift on the process output is amplified.

Again, the controlled process output is superior to that of the uncontrolled process. However, the effect of the shift on the controlled process output increases as the magnitude of the shift increases. Significant control is required to compensate these disturbances.

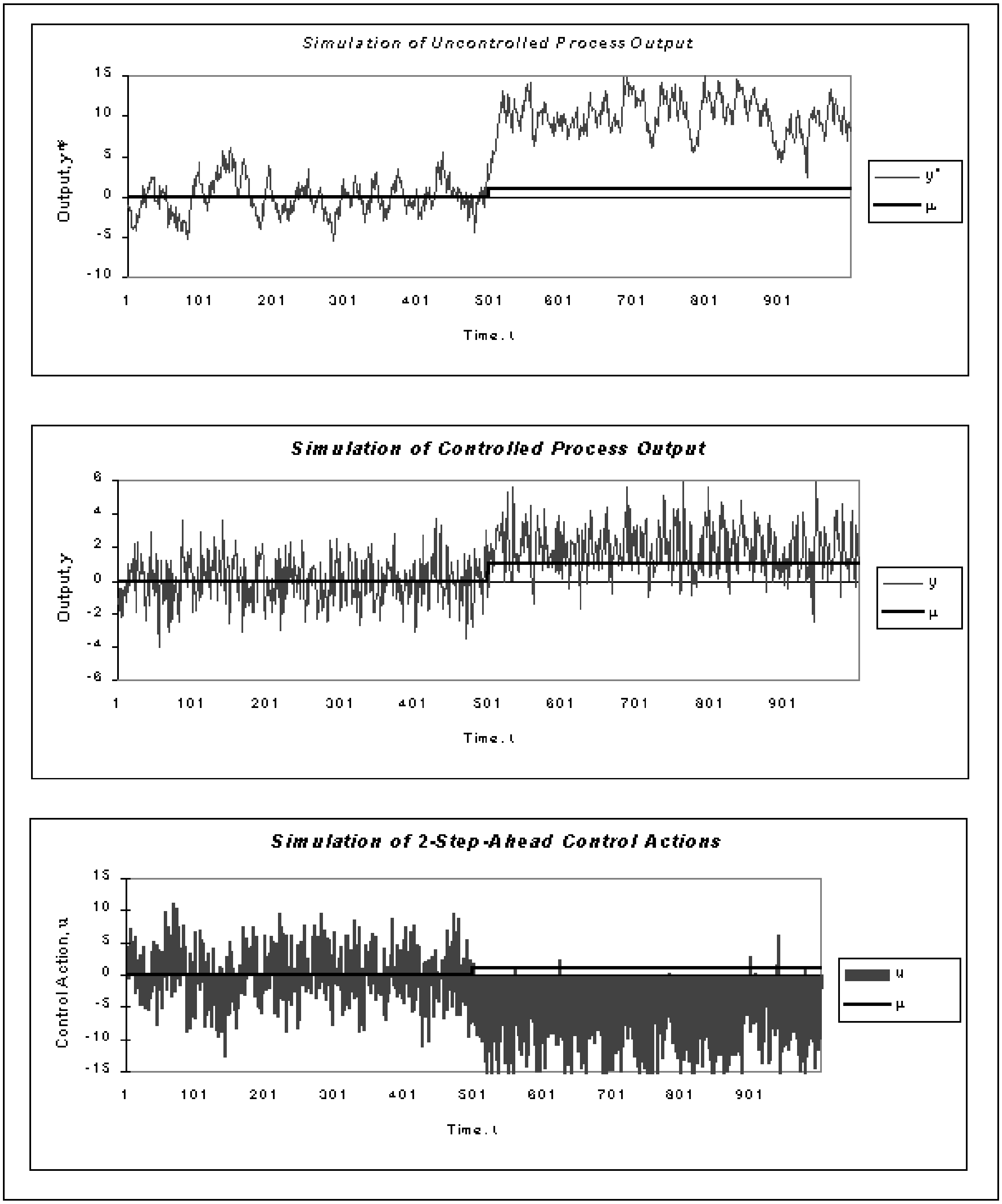

Figure 7: Simulation of the uncontrolled output, controlled output, and control signal of the system in Example 2 with $k = 2$ and the MV control strategy. A comparison between the uncontrolled and controlled outputs shows that the control strategy succeeds in reducing the output variation due to the disturbances process. However, the controller does not entirely reduce all of the variation due to the step-shift in period 500. Note particularly that following a 1 σ step-shift, the output of the controlled system varies about the mean of 1.9, which is the expected value of Eq. (25).

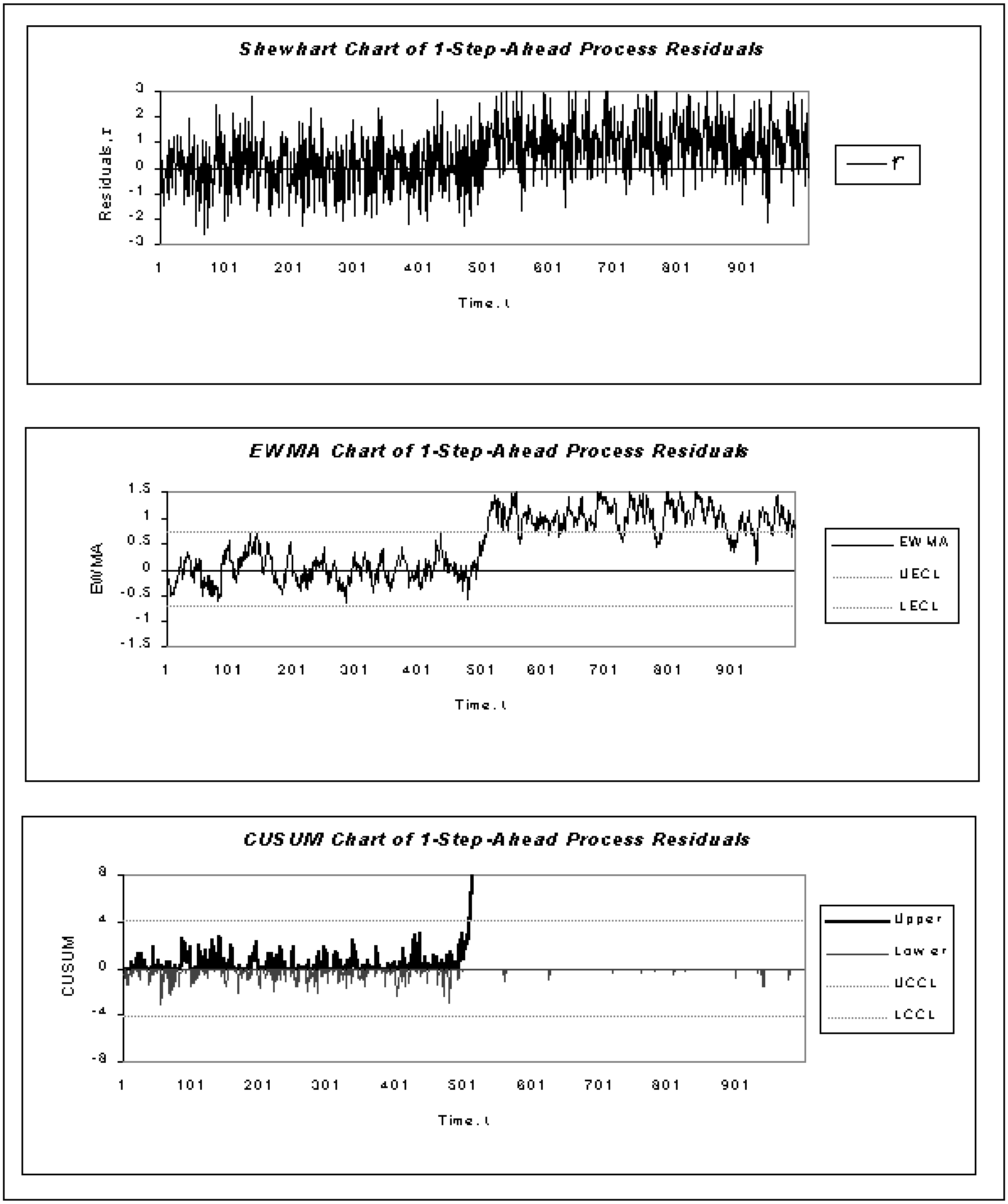

Figure 8: Shewhart, EWMA, and CUSUM charts of the residuals from the simulation results of Example 2. The standard Shewhart chart signals in period 535, the EWMA chart signals in period 509 and the CUSUM in period 510. As in Example 1, a 1 σ step-shift enters the system in period 500.

Method	$ARL_{1\sigma}$	$ARL_{2\sigma}$
Shewhart	44.1 (0.8)	6.5 (0.1)
EWMA	9.6 (0.1)	4.0 (0.04)
CUSUM	9.8 (0.1)	3.5 (0.04)

Table 2: Average Run Lengths for Example 2.

The standard Shewhart chart, the EWMA chart, and the CUSUM chart of the process residuals are shown in Figure 8. The Shewhart chart signals in period 535, 36 periods after shift onset. The EWMA chart of the residuals indicates a signal in period 509. As before, this chart was designed with $\rho = 0.133$ and $L = 2.777$, giving an "in-control" ARL of 370. The CUSUM chart detects the shift in period 510. Again, we choose $h = 4.1$ and $k = 0.5$ resulting in the same "in-control" ARL as that of the EWMA chart. These detection times agree nicely with our predicted ARLs shown in the third column of Table 1.

To generalize these results, a simulation study was again undertaken. 2500 experimental runs were performed as in Example 1 with a system described by Eq. 23. Parameter values remain the same and experiments were performed for two step-shifts, $1\sigma_\epsilon$ and $2\sigma_\epsilon$. The results are shown in Table 2.

Note that these results verify Eq. 24, our prediction for the behavior of the residuals.

Therefore, control charts may be designed without considering the effect of the disturbance dynamics on the process residuals. The SPC practitioner does not need to update his monitoring scheme and may expect it to behave as predicted in [3, 5, 6].

3.3 Shift Occurs Within the Underlying Noise Process

This case may be characterized by the following equation,

$$d_t = \lambda \frac{C'}{A^*}(\epsilon_t + \mu_t), \tag{27}$$

where μ_t is the value of the mean shift at time t. If we assume a system like Eq. 1, this situation may be represented by Figure 9. This type of shift could be an example of sudden and sustained changes in the environmental conditions affecting the output process.

The new system may be rewritten as,

$$\begin{aligned} y_t &= \frac{B'}{A'}u_{t-k} + \lambda \frac{C'}{A^*}(\epsilon_t + \mu_t) \\ &= \frac{B}{A}u_{t-k} + \lambda \frac{C}{A}(\epsilon_t + \mu_t). \end{aligned} \tag{28}$$

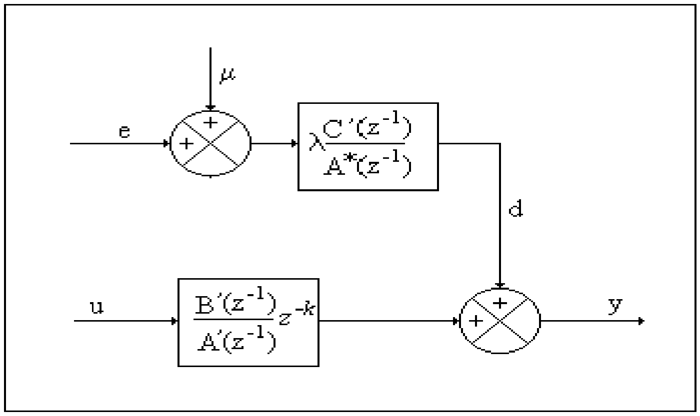

Figure 9: Model of shift entering dynamical system within the underlying noise process

Assuming that the system output is controlled with a MV controller and applying Eq. 3 and Eq. 4, we obtain the equation for the system output at time t,

$$y_t = \lambda F(\epsilon_t + \mu_t). \tag{29}$$

Again, the output behavior is affected by the presence of the process shift. The magnitude of this effect depends upon the coefficients of the polynomial F, and the form of the mean shift.

Combining Eq. 12 and Eq. 28, the effect of the shift on the process residuals is derived as,

$$r_t = y_t - \hat{y}_{t|t-1} = \lambda(\epsilon_t + \mu_t). \tag{30}$$

Thus, when a shift occurs within the underlying noise process and standard SPC techniques are applied to the process residuals, the ARLs will be exactly like those predicted. The ability of the residuals to display the effect of the shift is unchanged by the process dynamics and is independent of the process time delay k.

Example 3

Continuing the example of 4.1 with a mean shift occurring as in Eq. 27, we find that the control equation, output error, and residuals will be identical to those of Example 2. The result of the SPC monitoring will also be identical. This follows since, in Example 2, we have assumed a first-order autoregressive disturbance process ($C' = 1$), and an underlying white noise process having unit dispersion ($\lambda = 1$) in both. We can see from Eq. 30 that if $\lambda < 1$, the shift will appear to be diminished in the residuals. Conversely, if $\lambda > 1$, the shift will be exaggerated in the residuals.

3.4 Change in Dispersion

Suppose that a change in dispersion of the underlying noise process occurs in period t such that λ is replaces by λ^* in the system equation Eq. 2. Applying Eq. 3 and Eq. 4 results in the following regulation error in period t,

$$y_t = \lambda^* F \epsilon_t = \lambda^* \epsilon_t + \lambda(f_1 \epsilon_{t-1} + \ldots + f_{k-1}\epsilon_{t-k+1}).$$

The one-step-ahead forecasts are found using Eq. 6. From Eq. 7, the residuals are derived as,

$$\begin{aligned} r_t &= \lambda^* \epsilon_t + \lambda(f_1 \epsilon_{t-1} + \ldots + f_{k-1}\epsilon_{t-k+1}) - \lambda(f_1 \epsilon_{t-1} + \ldots + f_{k-1}\epsilon_{t-k+1}) \\ &= \lambda^* \epsilon_t. \end{aligned} \tag{31}$$

It can be shown that subsequent residuals in periods $(t+1), (t+2), \ldots$ will also take this form.

Therefore, the residuals are unaffected by the process dynamics when a change in dispersion occurs. Process monitoring techniques which use the residuals will detect the variance shift as well as would be detected if the process was i.i.d.

4. CONCLUSION

Several authors have recommended methods for monitoring controlled processes with standard SPC techniques. Most of these have focused on systems which take advantage of one-step-ahead MV controllers. The output of such a system will consist of a series of i.i.d. normal deviates. Therefore, using the output values as the plotted statistics produces SPC charts with predictable characteristics.

However, little attention has been paid to monitoring systems which utilize k-step-ahead MV controllers. We consider a system controlled by k-step-ahead MV controllers under three types of mean shifts and demonstrate that the mechanism by which upsets occur influences the ability of the residuals to detect the upsets. If shifts occur within the underlying noise process, standard SPC techniques will perform as predicted. However, if process upsets occur independently of the disturbance process, control chart performance might be affected. In these cases formulas have been provided which demonstrate how the residuals will be changed. We show that this effect is independent of the time delay, k. This information may be used to update control chart performance predictions or serve as an aid in modifying control chart design using results derived by other authors. We also show that the ability of the residuals to detect a change in the process dispersion is not altered by the disturbance dynamics.

Underlying this discussion has been the assumption that the SPC practitioner possesses a significant understanding of the process being studied. Effective application of the techniques described herein requires explicit knowledge of both the uncontrolled and the design of the k-step-ahead MV controller. This is not a serious problem if the monitoring scheme is designed in conjunction with the controller.

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