A price-setting newsvendor problem under mean-variance criteria

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Abstract

The single-product, single-period newsvendor problem with two decision variables, namely price and stock quantity, is considered. The performance measure, in addition to the expected revenue, includes the variance of the income scaled with a risk parameter. We present conditions for the concavity of this risk-sensitive performance measure and the uniqueness of the optimal solution for both risk-averse and risk-seeking cases under the additive demand model, and compare the results to others previously published. These conditions are introduced in terms of the lost sales rate elasticity. Furthermore, we provide numerical examples that aim to endorse the theoretical results herein explained.

Keywords: Inventory, Pricing, Revenue management, Risk analysis, Uncertainty modeling

1. Introduction

The newsvendor problem has been widely studied since it first appeared at the end of the *XIX* century. It is still the subject of further research that addresses more complex and realistic situations based on previous work. The problem, in its basic formulation, aims at finding an optimal replenishment policy of a perishable product in the face of an uncertain, stochastic demand. Such a solution is selected in a way that maximizes the expected profit, which is calculated as the difference between the expected income and the purchase cost of the good in question. Many modifications to this basic model have been proposed throughout the years, introducing additional complexity to the problem.

On the one hand, research efforts have focused on a profit-maximizer individual or business that wants to maximize the expected profit. The literature offers us a very diverse range of assumptions and approaches to different demand and inventory models. For example, Whitin [1] introduced the effect of price in the stochastic demand, thus adding this decision variable to the problem, on top of the stocking quantity. Petruzzi and Dada [2] used this price-demand relationship in several ways while presenting a single-period approach that maximized expected profit and established theorems that indicated how to select the stocking policy based on the statistical distribution that conferred the demand

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its stochastic nature. Such demand was presented in the form of an additive and in the form of a multiplicative model. Moreover, they presented a closed, analytical expression to determine the optimal price as a function of such stocking policy and compared it to the so-called riskless price as done in [3]. Also, they showed how pricing decisions affect demand uncertainty under various modeling assumptions. Federgruen and Heching [4] introduced a multi-period model for inventory control with backlogging which included a price-dependent stochastic demand and analyzed the optimal pricing and replenishment strategies to be set simultaneously in each period and how they compared when prices can be set bi-directionally or when only markdowns are allowed. Kocabiyikoglu and Popescu [5] unified and introduced the concept of *lost sales rate (LSR)* elasticity and explained the monotonicity of the optimal price and stock as a function of this new concept. Moreover, they set the properties that the LSR elasticity must possess for the profit to be jointly concave as a function of pricing and stocking decisions and for the problem to have a unique, optimal solution.

On the other hand, and as opposed to the norm in financial analysis, a tradeoff between expected return and risk in planning problems has been absent for many years. All the articles mentioned above presented a price-dependent demand and a profit-optimizer decision maker that is risk-neutral. This decision maker seeks to maximize the expected profit by finding an appropriate balance between expected income and expected costs. However, these models do not take into account the variance of the income, and therefore it is likely to select an optimal policy that maximizes the expected income but presents a variability that turns this decision into a risky bet. Some research efforts have taken place with respect to this approach. However, albeit the demand keeps its stochastic nature, in many cases it is not presented as a function of the price and therefore the optimization is sought by means of only selecting an optimal quantity of product to purchase, thus disregarding pricing decisions. To the best of our knowledge, it was Lau [6] who first considered a mean-standard deviation payoff criterion within the newsvendor model, although he only gave the equation whose root provided the optimal quantity. Chen and Federgruen [7] presented this mean-variance analysis for different planning problems, namely, the newsvendor problem, the base stock problem and the *(R, nQ)* model. When dealing with the newsvendor problem, they applied a single-stage model that optimized a utility function for risk-neutral, risk-averse and risk-seeking decision maker over a feasible region given by an efficient frontier. Later on, Choi *et al.* [8] introduced stockout costs in the mean-variance analysis and presented results for various risk attitudes under demands that followed well-known statistical distributions. Wu *et al.* [9] focused on the impact that stockout costs have on the optimal ordering decisions when comparing classic models and mean-variance analysis models, showing via numerical results that, for a given stockout price, a mean-variance analysis yields a lower optimal order quantity and a lower optimal value of the problem. Özler *et al.* [10] offered a one-stage, multi-product approach with value-at-risk considerations that included mathematical programming results for the case of one and two products and an approximation to the *N*-product case. Wang and Webster [11] introduced an alternative approach by using a piece-wise linear loss-aversion utility function. In [12] Wang *et al.* studied the newsvendor problem within the expected utility framework and considered three different utilities, namely, constant absolute risk aversion (CARA), decreasing absolute risk aversion (DARA), and increasing absolute risk aversion (IARA), whereas Choi and Ruszczyński [13] examined this model with an exponential utility function used to model a risk-averse decision-maker with a constant risk coefficient in the sense of the Arrow-Pratt measure.

In the present paper, our aim is to combine the two approaches introduced above. That is, we present a mean-variance analysis of the newsvendor problem that includes a stochastic, price-dependent demand. This problem was presented before in [14], but it was approached from the perspective of the *expected utility framework*, selecting concave utility functions to model risk-averse situations and finding the relation of optimal pricing and stocking strategies with respect to the levels of risk aversion. There have been other similar efforts; in particular, Chen *et al.* [15] proposed a model based on a conditional value-at-risk decision criterion (CVaR), where it was analyzed how the optimal price and stock quantity changed when varying the $\eta - quantile$ of the function they sought to maximize. Our work is focused on an alternative approach that penalizes the variability of the profit in the objective function and rests in earlier research done in [16] and [17] when tackling Markov decision processes. On this basis, we perform such analysis for an additive demand model under several risk assumptions, in order to yield results that allow us to discern how the newsvendor's risk attitude affects his optimal stocking and pricing policies. Additionally, these results are extended to the field of economics by providing an interpretation in terms of the (LSR) elasticity.

The organization of this paper is as follows. In §2, we introduce the problem formulation. In §3, we focus on the risk-averse environment and set forth the assumptions under which the risk-sensitive performance measure can be both sequentially optimized and simultaneously optimized. We also provide insight about the behavior of the optimal pricing and stocking decisions as a function of the risk aversion, and show numerical examples where these results can be visually analyzed. In §4 we approach the problem again under a risk-seeking environment and state similar conditions under which the performance measure presents a predictable behavior. Finally, we summarize our conclusions in §5.

2. Problem formulation

Consider a decision maker who sells a perishable product, and has the ability to decide on the quantity to produce or buy and the price to set for the good he or she sells. Moreover, such a decision must be based on the expected revenue for the upcoming period, the production/procurement costs and the variability of the revenue. Assuming that the demand for a period is random and a function of the price we introduce the following risk-sensitive performance measure

$$\tilde{P}(p, x) \quad = \quad pE(min\{D(p, \epsilon), x\}) - cx - \lambda Var(p \cdot min\{D(p, \epsilon), x\}), \tag{1}$$

where p is the price set for the good and ϵ is a continuous random variable with expected value $E(\epsilon \geq 0)$ and finite variance $Var(\epsilon)$. This random variable with support $[A, B]$, where $A < 0$ and $B > 0$, has a probability distribution given by the continuous density function f and the cumulative distribution function F. We assume that F is twice differentiable everywhere on its domain. Furthermore, x is the inventory level set for the good, c is the production/procurement cost per unit of finished product and $D(p, \epsilon)$ is the demand for a single period given the price p and the realization of the random variable ϵ. Finally, λ is a risk parameter greater than 0 for risk-averse cases, smaller than 0 for risk-seeking cases and equal to 0 in risk-neutral cases.

Although the risk-sensitive performance measure defined in eq.(1) lacks economic meaning, optimizing eq.(1) must be understood as maximizing profit while minimizing variance. Furthermore, the parameter λ can be considered as a scaling factor that balances the expected profit and the variance of the profit. In addition, the expression above contains several assumptions. First, it assumes the procurement costs increase linearly with the quantity bought. Implicitly, this means that batch production is not more convenient economically than item-by-item production and the economies of scale do not apply. The introduction of the variability of the revenue weighted with the parameter λ is a different approach compared to that presented in [7], that presented the attitude towards risk modeled by an utility function that was concave, convex and linear for risk-averse, risk-seeking and risk-neutral settings, respectively. It is also different from CVaR-based models as shown in [15]. Finally, we introduce the demand $D(p, \epsilon)$ in the same fashion as [18]:

$$D(p, \epsilon) \quad = \quad g(p)\epsilon + y(p),$$

where both $g(\cdot)$ and $y(\cdot)$ model nonincreasing functions of p. In the present article, we assume that $g(p) \equiv 1$ and $y(p) \equiv a - bp$ with $a, b > 0$ and therefore we deal with an additive demand model of the form

$$D(p, \epsilon) \quad = \quad a - bp + \epsilon,$$

which includes a non-random term, usually referred to as *riskless demand* ([3]). Furthermore, this additive model implies that pricing decisions do not affect the variability of the demand, as $Var[D(p, \epsilon)] = Var(\epsilon)$.

3. Risk-Averse Newsvendor

The decision maker seeks to maximize the performance measure (1). We set forth the following assumptions:

(A1) $p \in (c, p_{max}]$ where $p_{max} \leq \dfrac{a}{b}$, and $y(p) = 0,\ \forall p \notin (c, p_{max}]$,

(A2) $\dfrac{a + E[\epsilon]}{b} - p_{max} \leq p_{max} - c$,

(A3) $\lambda < \dfrac{1}{4(B - E(\epsilon))p_{max}}$,

(A4) $A + y(c) > 0$.

(A1) indicates that the price to be set has to be bigger than the unit cost of production and smaller than p_{max}. The latter will never be greater than the price at which the riskless demand for the product, $y(p)$, equals 0. When $y(\cdot)$ is a linear function, this value turns out to be $\frac{a}{b}$. (A2) imposes that p_{max} will always be at least as close to $\frac{a + E[\epsilon]}{b}$ as it is to c. The third assumption will become important in the subsequent analysis and the last assumption imposes that regardless of how small A is, the lowest price that can be set guarantees that the realization of the demand, $D(p,\epsilon)$, will still be positive.

Let us define $z = x - y(p)$. We can transform $\tilde{P}$ in (1) into a function of (p,z) as follows:

$$\begin{aligned}\tilde{P}(p,x) &= pE(\min\{\epsilon,z\}) + py(p) - c(z+y(p)) - \lambda Var(p\min\{\epsilon,z\}) \\ &= p\mu(z) - \lambda p^2\sigma^2(z) + py(p) - c(z+y(p)) =: P(p,z),\end{aligned}$$

where

$$\begin{aligned}\mu(z) &= E(min\{\epsilon,z\}) = & E(\epsilon) + \int_z^B (z-u)f(u)\,du, \quad z\in[A,B], \\ \sigma^2(z) &= Var(min\{\epsilon,z\}) = & Var(\epsilon) + \int_z^B (z^2-u^2)f(u)\,du - \left[\int_z^B (z-u)f(u)\,du\right]^2 \\ && -2E(\epsilon)\int_z^B (z-u)f(u)\,du, \quad z\in[A,B].\end{aligned}$$

As indicated in [2], for a selected value of z we face shortages if $\epsilon > z$ and leftovers if $\epsilon < z$. A very immediate interpretation of z is that of safety stock, for it is defined as the difference between the actual stock level and expected demand. Understanding the behavior of the functions above is crucial for the analysis that will be shown later. On the one hand, $\mu(\cdot)$ is always an increasing function of z in $[A,B]$, for $\mu(A) = A < 0$, $\mu(B) = E(\epsilon)$ and $\frac{d\mu(z)}{dz} = 1 - F(z)$. On the other hand, $\sigma^2(\cdot)$ is a nonnegative, increasing function of z, with $\sigma^2(A) = 0$, $\sigma^2(B) = Var(\epsilon)$ and $\frac{d\sigma^2(z)}{dz} = 2[1-F(z)][z-\mu(z)] \geq 0$. After further simplifications, the decision-maker's problem can be written as:

$$\max_{p,z} P(p,z) = \max_{p,z} \left(-p^2[\lambda\sigma^2(z)+b] + p[\mu(z)+a+cb] - c(z+a)\right). \tag{2}$$

3.1. Sequential optimization

Sequential optimization seeks optimization of a function of several variables by sequentially selecting the optimal values of each variable that will, at the end, produce the maximum of the function that we need to maximize. Zabel [19] proposes a method by which it is possible to find the optimal price that maximizes the performance measure for a given z. Then, this function can be expressed in terms of only one variable, z, and consequently optimized. However, finding the optimal price requires concavity of p with respect to z. To this end, we have

$$\frac{\partial P(p,z)}{\partial p} = -2p(\lambda\sigma^2(z)+b) + (\mu(z)+a+cb), \tag{3}$$

$$\frac{\partial^2 P(p,z)}{\partial p^2} = -2(\lambda\sigma^2(z)+b). \tag{4}$$

The risk setting, $\lambda > 0$, and the nonnegativity of b and $\sigma^2(z)$ defines the performance measure $P(\cdot)$ such that $P(\cdot,z)$ is concave, since $\frac{\partial^2 P(p,z)}{\partial p^2} < 0$. Forcing (3) to be equal to 0 yields the price p that maximizes $P(\cdot,z)$ for a given z:

$$p^*(z) = \frac{\mu(z)+a+cb}{2[\lambda\sigma^2(z)+b]}. \tag{5}$$

Lemma 1. *The optimal price p^* is uniquely determined by (5) and is increasing. Moreover, $p^*(z) \in (c, p_{max}]$, for any $z \in [A, B]$.*

Proof. The optimal price at $z = A$ is greater than the cost c, for $p^*(A) = \dfrac{A + a + cb}{2b} > c$. On the other hand, an upper bound of $p^*(\cdot)$ is given by

$$p^*(z) \leq \quad \frac{E(\epsilon) + a + cb}{2b} \quad \leq p_{max},$$

which is guaranteed by assumption (A2). It remains to prove that $p^*(\cdot)$ is increasing. Indeed:

$$\frac{dp^*(z)}{dz} \quad = \quad \frac{1 - F(z)}{2[\lambda\sigma^2(z) + b]}[1 - 4\lambda(z - \mu(z))p^*(z)]. \tag{6}$$

The expression above is always positive (i.e. $p^*(\cdot)$ is increasing) provided that $1 - 4\lambda(z - \mu(z))p^*(z) > 0$. However, we know that

$$1 - 4\lambda(z - \mu(z))p^*(z) \geq 1 - 4\lambda(B - E(\epsilon))p_{max},$$

whence we obtain the condition that for the expression above to be positive we need:

$$\lambda \quad < \quad \frac{1}{4(B - E(\epsilon))p_{max}},$$

which was assumed by (A3). Therefore, $p_{max} \geq p^*(z) > c, \ \forall z \in [A, B]$. □

Remark 1. *The optimal price for a given z, $p^*(z)$, is smaller in the risk-averse case ($\lambda > 0$) than in the risk-neutral case ($\lambda = 0$). This conclusion is correct both mathematically and intuitively, for a risk-averse individual will set lower prices to make sure that sales are as high as possible, and endorses the results obtained in [14]. The price in the risk-neutral case is in turn smaller than or equal to the optimal riskless price as observed in [2]. However, the results in [14] claim that the optimal risk-neutral price equals the optimal riskless price, whereas the model proposed in this paper suggests that the optimal risk-neutral price is in between the optimal risk-averse price and the optimal riskless price.*

Definition 1. *The risk-sensitive performance measure under the best price function of the safety stock z is defined as*

$$P^*(z) := P(p^*(z), z) = \quad \frac{1}{4}\frac{(\mu(z) + a + cb)^2}{\lambda\sigma^2(z) + b} - c(z + a) \quad = \frac{1}{2}p^*(z)(\mu(z) + a + cb) - c(z + a), \tag{7}$$

Its first derivative with respect to z is given by

$$\frac{dP^*(z)}{dz} \quad = \quad p^*(z)(1 - F(z))[1 - 2\lambda(z - \mu(z))p^*(z)] - c. \tag{8}$$

Definition 2. *([5])The lost sales rate (LSR) elasticity for a given price p and inventory level x is defined as*

$$\tilde{\kappa}(p, x) \quad = \quad \frac{p(G(p, x))'_p}{1 - G(p, x)},$$

where $G(p, x) := Pr(D(p, \epsilon) \leq x)$ and $(G(p, x))'_p \equiv \dfrac{\partial G(p, x)}{\partial p}$.

Furthermore, for the additive case we know that

$$Pr(y(p) + \epsilon \leq x) = \quad Pr(\epsilon \leq x - y(p)) \quad = F(z).$$

Hence, by means of a direct application of *Leibniz's Rule* we can further simplify the expression above and write it in terms of z as shown below:

$$\tilde{\kappa}(p,x) = \quad \frac{p(G(p,x))'_p}{1 - G(p,x)} = \frac{pbf(z)}{1 - F(z)} \quad =: \tilde{\varepsilon}(p,z).$$

Moreover, by means of *Lemma* 1, we can introduce

$$\tilde{\varepsilon}^*(z) \quad := \quad \tilde{\varepsilon}(p^*(z), z).$$

Theorem 1. *Assume that*

$$\tilde{\varepsilon}^*(z) := \quad \frac{bp^*(z)f(z)}{1 - F(z)} \quad \geq \frac{1}{2}.$$

Then, the single-period optimal stocking and pricing policy for the case of additive demand is to stock $x^* = y(p^*) + z^*$ *units to sell at the unit price* p^**, where* p^* *is specified by Lemma* 1 *and* z^* *is the unique root of the equation*

$$p^*(z)(1 - F(z))[1 - 2\lambda(z - \mu(z))p^*(z)] - c \quad = \quad 0.$$

Proof. See Appendix E. □

Remark 2. *The result shown by Theorem 1 matches that found in [5] for a risk-neutral individual (*$\lambda = 0$*).*

Albeit we have analyzed the difference between the price $p^*(z)$ that maximizes the performance measure for a given value of z in risk-neutral and risk-averse environments, it remains to see what happens to $z^*(p)$, the value of z that maximizes this measure for a given price p. The first and second partial derivatives of $P(\cdot)$ with respect to z as well as the cross partial derivative yield

$$\frac{\partial P(p,z)}{\partial z} \quad = \quad p(1 - F(z))[1 - 2\lambda p(z - \mu(z))] - c, \tag{9}$$

$$\frac{\partial^2 P(p,z)}{\partial z^2} \quad = \quad pf(z)[2\lambda p(z - \mu(z)) - 1] - 2\lambda p^2 F(z)(1 - F(z)), \tag{10}$$

$$\frac{\partial^2 P(p,z)}{\partial p \partial z} \quad = \quad [1 - F(z)][1 - 4\lambda p(z - \mu(z))]. \tag{11}$$

A closer look to the formulae above reveals that in a risk-neutral setting $P(p,\cdot)$ is concave for a given price p and the maximum of the performance measure is obtained at $z = F^{-1}(1 - \frac{c}{p})$, which is a well-known result. When $\lambda > 0$ this objective function is still concave, for (A3) guarantees that $2\lambda p(z - \mu(z)) - 1$ is negative. However, there is not a closed form that yields the optimum value z^*, which solves the equation

$$(1 - F(z))[1 - 2\lambda p(z - E(\epsilon) - \int_z^B (z - u)f(u)\,du)] - \frac{c}{p} \quad = \quad 0. \tag{12}$$

Let us fix $p \in (c, p_{max}]$. In the following lemma we examine the dependence of z^* on λ. This dependence is expressed by $\tilde{z}^*(\lambda)$ to indicate the value of z^* at a given point p for some λ.

Lemma 2. *The function* $\tilde{z}^*(\cdot)$ *is decreasing.*

Proof. See Appendix E. □

Remark 3. *The result above endorses that obtained under CVaR considerations in [15] and also that yielded by the use of the expected utility framework in [14]. In that sense, the model proposed here provides certainty with respect to the behavior of the optimal order quantity in additive models of the form* $a - bp + \epsilon$ *in the face of risk-averse environments.*

3.2. Simultaneous optimization

Unlike we proceeded in §3.1, we focus now on giving conditions to jointly optimize price and quantity decisions simultaneously, thus guaranteeing that (2) has a unique solution.

Theorem 2. *If* $\tilde{\varepsilon}(p,z) \geq \frac{1}{2}$, *then* $P(\cdot)$ *is jointly concave in* p *and* z *and the problem referenced by (2) has a unique price-quantity solution* $(p^*, z^* + y(p^*))$.

Proof. See Appendix E. □

Remark 4. *The result shown by Theorem 2 also matches that found in [5] for a risk-neutral individual* ($\lambda = 0$).

Notice that this condition for joint concavity is very restrictive as it requires the LSR elasticity to be greater or equal than $\frac{1}{2}$ in the whole domain of the function under study. This condition is sufficient to guarantee that (p^*, z^*) is a maximum, but it is not necessary. If the function were not jointly concave, the state of (p^*, z^*) as a critical point would not be altered, for the only necessary and sufficient condition for criticality is that the Hessian matrix is negative semidefinite at that precise point and this is guaranteed if $\tilde{\varepsilon}^*(z) \geq \frac{1}{2}$, which is in turn a sufficient condition for the existence of a unique maximum in $P(\cdot)$ [20].

3.3. Numerical examples

In order to illustrate the ideas previously exposed, we proceed with several numerical examples using different distribution functions for the random variable ϵ. In particular, we present ϵ as a random variable uniformly distributed in $[A, B]$, and a random variable normally distributed with mean $\mu = 0$ and standard deviation $\sigma = 10$, and truncated below A and above B. For both cases, we use the following parameters to define the problem: $A = -10$, $B = 10$, $a = 35$, $b = 1$, $c = 10$, and $\lambda_{max} < \frac{1}{4Bp_{max}} \leq \frac{1}{1400}$. In addition, we include the case of a uniform distribution with expectation different from 0 with $A = -3$, $B = 40$, $a = 35$, $b = 1.5$, $c = 10$, and $\lambda_{max} < \frac{1}{4(B - E(\epsilon))p_{max}}$. Per Assumption (A2), $p_{max} \geq 22.5$ in the first two cases and $p_{max} \geq 23.83$ in the case of the uniform distribution with expectation different from 0. *Table A1* contains, for these three distributions, the optimal values of p, z and $P(p,z)$ for each value of λ that was put to the test. All tables and figures referenced can be found in Appendix A and Appendix C, respectively.

With respect to the uniform distribution centered at 0, *Figure C1* shows the optimal price $p^*(\cdot)$ and its first two derivatives for different values of λ that range from λ_{max}, representing the highly risk-averse newsvendor, to 0, representing a risk-neutral newsvendor. In this case, the impact of risk-aversion on the optimal price $p^*(\cdot)$ is not very significant, being of 2.32% when $z = 10$ but it is interesting to verify that increasing risk-aversion leads to smaller optimal prices and how this price is concave in $[A, B]$. Also, *Figure C2* shows how the LSR elasticity $\tilde{\varepsilon}^*$ is always greater than $\frac{1}{2}$, which in turn guarantees that $P^*(\cdot)$ is concave. Note that since $p^*(\cdot)$ differs very little for different values of λ, the curves of $\tilde{\varepsilon}^*(\cdot)$ almost overlap. Also, the optimal value of the objective function P^* is inversely proportional to the value of λ and its optimum value changes only 3.45% between the most risk-averse situation and the risk neutral situation. It is worth mentioning that in this case $\tilde{\varepsilon}(p, z) \geq \dfrac{1}{2}$ and therefore the performance measure is jointly concave.

In the case of a truncated normal distribution, when comparing the most risk-averse situation with the risk-neutral case, the optimal price p^* reveals a difference of 2.44% at $z = 10$, which is very similar to that is found in the previous numerical example. Again, and endorsing the theoretical results, $p^*(\cdot)$ is concave and $\tilde{\varepsilon}^*(z) \geq \dfrac{1}{2}$, which guarantees the existence of a unique optimum. Such optimum yields a gap of 3.02% between the most risk-averse case and the risk-neutral scenario. This difference is as well very similar to that is shown for the uniform distribution (3.02% vs. 3.45%). Unlike the uniform case, this distribution does not provide $P(\cdot)$ with joint concavity, as there are some values of $\tilde{\varepsilon}(p, z)$ under $\dfrac{1}{2}$.

Finally, the uniform distribution with mean different from 0 yields a bigger gap of the optimal price at the right extreme of the interval, with this price at the risk-neutral case being 5.12% greater than in the most-risk averse case for $z = B = 40$.The optimal lost sales rate elasticity remains greater than $\dfrac{1}{2}$ at all times, which produces a concave curve for $P^*(\cdot)$ that has a more significant variation between the risk neutral case and the most risk-averse situation (8.44%). Like in the case of the truncated normal distribution, $\tilde{\varepsilon}(p, z)$ is smaller than $\dfrac{1}{2}$ in some regions, which leads to a surface that is not jointly concave.

We also calculate the values of $\tilde{z}^*(\lambda)$ for different λ and a given price $p = 20$, using the same parameter values specified at the beginning of this section. The results can be found on *Table A2* and show, as claimed by *Lemma 2*, that the risk-aversion is inversely proportional to the optimum value of $z^*(20)$.

In order to show the effect that risk aversion has on the profit, *Table A1* and *Table A2* also include the expected profit and the standard deviation of the profit for each case considered. These can be calculated directly from (1). Intuitively, the expected profit and the standard deviation of the profit should decrease as the level of risk aversion increases. Indeed, this is the case in all the different scenarios presented.

4. Risk-seeking newsvendor

Whereas the expressions of all the partial derivatives previously obtained are still valid for risk-seeking situations, the conditions under which the performance measure is concave are greatly modified by the fact that $\lambda < 0$. We turn now our attention to this case in which, again, we are seeking the maximization of

$$P(p,z) \quad = \quad p\mu(z) - \lambda p^2\sigma^2(z) + py(p) - c(z + y(p)).$$

Furthermore, we set forth the following assumptions, some of which were explained formerly (we refer the reader to §3 for the justification of the first three of them). The fourth assumption will become relevant in the next subsection. Note that in this case we have not included any assumption on the value of λ; this condition will be addressed as part of the analysis that will follow.

(B1) $p \in (c, p_{max}]$ where $p_{max} \leq \frac{a}{b}$, and $y(p) = 0 \; \forall p \notin (c, p_{max}]$,

(B2) $\frac{a + E(\epsilon)}{b} - p_{max} \leq p_{max} - c$,

(B3) $A + y(c) > 0$,

(B4) $E(\epsilon) < y(c)$.

4.1. Sequential optimization

Lemma 3. *The risk-sensitive performance measure $P(\cdot, z)$ is concave for each z, if $\lambda \geq \frac{-b}{Var(\epsilon)}$.*

Proof. We proceed as in the risk-averse case. However, the inequality $\frac{\partial^2(p,z)}{\partial p^2} = -2(\lambda\sigma^2(z) + b) < 0$ does not hold for every value of λ. Therefore, setting $\lambda \geq \max_z \frac{-b}{\sigma^2(z)}$ solves this problem and, given that $\sigma^2(\cdot)$ is an increasing function, it is enough to set $\lambda \geq \frac{-b}{\sigma^2(B)} = \frac{-b}{Var(\epsilon)}$. □

The result shown by *Lemma* 3 may not respect (B1). In fact, as shown analytically in *Lemma 1*, for $\lambda = \frac{-b}{Var(\epsilon)}$ the price $p^*(z)$ goes to infinity and violates the assumption that this quantity cannot be greater than p_{max}. For that reason, we need to further restrict the possible selection of values for λ.

Lemma 4. *For a fixed z and a value of λ in the interval $\left[\frac{b(E(\epsilon) - y(c))}{2aVar(\epsilon)}, 0\right)$ the optimal price is determined uniquely as a function of z as shown by (5) and its value is always contained in $(c, p_{max}]$.*

Proof. See Appendix E. □

Remark 5. *In the risk-seeking case ($\lambda < 0$), the optimal price for a given z, $p^*(z)$, is greater than that is found for the risk-neutral case ($\lambda = 0$). This conclusion is correct both mathematically and intuitively, for a risk-seeking individual will set higher prices in his pursuit of greater profit, accepting the risk of selling less as a result of such decision.*

Once the conditions for $P(\cdot, z)$ to be concave is established, analogous to what was shown for the risk-averse case, we address the concavity of $P^*(\cdot)$.

Theorem 3. *Assume that for every $z \in [A, B]$*

$$\tilde{\varepsilon}^*(z) \quad \geq \quad b\frac{dp^*(z)}{dz} - \frac{2\lambda p^*(z)b\left[F(z)p^*(z) + (z-\mu(z))\frac{dp^*(z)}{dz}\right]}{1-2\lambda(z-\mu(z))p^*(z)}. \tag{13}$$

Then the single-period optimal stocking and pricing policy for the case of additive demand is to stock $x^* = y(p^*) + z^*$ units to sell at the unit price p^*, where p^* is specified by Lemma 4 and z^* is the unique root of equation (9).

Proof. See Appendix E. □

Obviously this condition for concavity requires evaluating the function for any $z \in [A, B]$. There exist, however, more expeditive approaches to rule out concavity: given any distribution, equation (13) shows that $\tilde{\varepsilon}^*(A) \geq \frac{1}{2}$ and $\tilde{\varepsilon}^*(B) \geq \frac{-2\lambda b p^{*^2}(B)}{1-2(B-E(\epsilon))p^*(B)\lambda}$. This means that if we find the optimal LSR elasticity values at the extreme of the intervals to be less than these quantities, $P^*(\cdot)$ will not be concave. Also, by the definition of the LSR elasticity, if the chosen distribution has an increasing hazard rate, so does its optimal elasticity. This means that if $\tilde{\varepsilon}^*(z) < \frac{1}{2}$ for any given z, then its value at $z = A$ will not be greater or equal to $\frac{1}{2}$ and therefore $P^*(\cdot)$ will not be concave.

In an effort to come up with a friendlier condition for the concavity of $P^*(\cdot)$, we can estimate an upper bound of the right hand side of (13) and set

$$\tilde{\varepsilon}^*(z) \quad \geq \quad bK - 2\lambda a\left(\frac{a}{b} + (B - E(\epsilon))K\right), \tag{14}$$

with K being the maximum value that $\frac{dp^*(z)}{dz}$ attains in $[A, B]$. A priori, if the behavior of the function $z \mapsto \frac{dp^*(z)}{dz}$ is unknown, this maximum value can be large, with a rough upper bound given by $\frac{1-4\lambda(B-E(\epsilon))\frac{a}{b}}{2(\lambda Var(\epsilon)+b)}$, and therefore this expression might not be effective. If, like it was the case in risk-averse situations, this function is decreasing (i.e., $p^*(\cdot)$ is concave), then it attains a maximum value of $\frac{1}{2b}$ at $z = A$ and the bound above becomes

$$\tilde{\varepsilon}^*(z) \quad \geq \quad \frac{1}{2} - \lambda\frac{a}{b}(2a + B - E(\epsilon)).$$

This is potentially a very useful result, but requires the concavity of $p^*(\cdot)$. In what follows, we show the conditions that are needed for such a case to take place.

Lemma 5. *If*

$$\lambda \quad \geq \quad \max_z \left\{\frac{-f(z)b}{4(a+B-E(\epsilon)) + f(z)Var(\epsilon)}\right\}, \quad z \in [A, B],$$

then the function $z \mapsto \frac{dp^*(z)}{dz}$ is decreasing.

Proof. See Appendix E. □

The previous result allows us to state the following theorem.

Theorem 4. *Assume that*

$$\lambda \quad \geq \quad \max_z \left\{ \frac{-f(z)b}{4(a+B-E(\epsilon)) + f(z)Var(\epsilon)} \right\}, \quad z \in [A, B],$$

and

$$\tilde{\varepsilon}^*(z) \quad \geq \quad \frac{1}{2} - \lambda \frac{a}{b}[2a + B - E(\epsilon)].$$

Then the single-period optimal stocking and pricing policy for the case of additive demand is to stock $x^* = y(p^*) + z^*$ units to sell at the unit price p^*, where p^* is specified by Lemma 4 *and z^* is the unique root of equation (9).*

Proof. Since λ is contained in a range that guarantees that the function $z \mapsto \frac{dp^*(z)}{dz}$ is decreasing, then we have that $K = \left.\frac{dp^*(z)}{dz}\right|_{z=A} = \frac{1}{2b}$ and therefore (14), which establishes a condition for the concavity of $P^*(\cdot)$, can be written as

$$\tilde{\varepsilon}^*(z) \quad \geq \quad \frac{1}{2} - \lambda \frac{a}{b}[2a + B - E(\epsilon)].$$

By virtue of (E.2) and (E.3) there exists a point $z^* \in (A, B)$ at which the function $P^*(\cdot)$ attains a maximum, and such point is uniquely determined by the root of equation (9). □

Remark 6. *Given an appropriate range of values for λ, the LSR elasticity that is required in risk-seeking cases is greater than that is required in risk-averse situations.*

Lemma 6. *Define*

$$\hat{\varepsilon}^*(p) \quad = \quad \tilde{\varepsilon}(p, z^*(p)),$$

and let p_A and p_B be prices in the interval $(c, p_{max}]$ which yield $z^*(p_A) = A$ and $z^*(p_B) = B$, respectively, and that may or may not exist. Then, if

$$\hat{\varepsilon}^*(p) > bp\frac{1}{B - E(\epsilon)}, \quad c < p \leq p_{max},$$

the optimal safety stock $\tilde{z}^*(\cdot)$ is a decreasing function of λ, i.e., given a price p, the optimal safety stock increases as we face more risk-seeking situations.

Proof. See Appendix E. □

4.2. Numerical Examples

We proceed now to present some examples for the risk-seeking case. As the reader may have noticed, this case is more intrincated than the risk-averse case, with more conditions to be satisfied for obtaining a desirable behavior of the performance measure. Our experience has been that selecting λ according to *Theorem 3* induces a more significant impact on the optimal value of the objective function. These examples gave way to a larger range of values for the risk parameter, as we only require that $\lambda \in \left[\frac{b(E(\epsilon) - y(c))}{2aVar(\epsilon)}, 0\right)$. As far as the optimal elasticity, $\tilde{\varepsilon}^*(z)$, *Theorem 3* provides a more complex expression than *Theorem 4* that needs to be evaluated at any $z \in [A, B]$. Despite this hindrance, it was not particularly difficult to find instances where this condition was satisfied and allowed us to get significant results. All the tables and figures referenced in the next lines can be found in Appendix B and Appendix D respectively.

We present the case that ϵ is uniformly distributed in the range $[-30, 200]$ with demand given by $D(p, \epsilon) = 600 - 60p + \epsilon$. The purchase cost of the item we sell is $c = 7$. It can be seen in *Figure D1* how optimal price $p^*(\cdot)$ may not be concave for some values of λ but still is an increasing function of z and its value is held between $c = 7$ and $p_{max} = a/b = 10$ as expected whenever λ is greater than or equal to $\frac{b(E(\epsilon) - y(c))}{2aVar(\epsilon)}$. The risk-neutral case yields as well an optimal price which is 8.5% smaller than that of the most risk-seeking setting. *Figure D2* shows for the most risk-seeking case that $\tilde{\varepsilon}^*(\cdot)$ is always above the necessary and sufficient condition expressed by eq.(13), thus conferring concavity to the performance measure for this value of λ. For clarity, we omit similar curves for other tested scenarios but, as the fourth chart in this figure shows, all cases satisfy (13). In this example, there is a considerable gap in the optimal value of the objective function between a risk-neutral scenario and the most risk seeking case, with the latter being 22.66% smaller than the former.

The last example introduces a normal distribution with mean 25 and variance 1600 truncated below by -30 and above by 100. The demand follows the expression $D(p, \epsilon) = 175 - 35p + \epsilon$ where ϵ represents the random variable with the said distribution. The purchase cost of the item that we sell is now $c = 2.7$. These parameters produce a noteworthy effect on the objective function that is even more remarkable than that in the last example: the risk-neutral case yields an optimum value which is 43.80% lower than that of the most risk-seeking scenario. Likewise, the optimal price, is 15.3% lower when $\lambda = 0$ than when it is at its minimum value. Albeit $p^*(\cdot)$ is not concave in this case either, the condition for the concavity of $P^*(\cdot)$, as established in *Theorem 3*, holds for all λ considered. *Table B1*, in turn, displays the optimal pricing and safety stock decisions along with the optimum value of the objective function, the expected profit, and the standard deviation of the profit that they produce for the different values of λ that were attempted. Note that in both cases the impact of risk-seekingness in the nature of the profit as a stochastic variable is remarkable, to the point that, when $\lambda = \lambda_{min}$, the decision-maker faces scenarios where he or she must expect a loss in exchange for a much greater variability in the profit.

5. Conclusions

The approach that our model intends to give to the newsvendor problem aims at serving as a comparison with previously used models based on CVaR measures and the expected utility framework. Here we introduce a simple yet powerful variation of the single-period, price-dependent demand newsvendor problem with two decision variables (namely, price and stock quantity) by including the variability of the demand scaled by the attitude towards risk that the seller has. Such attitude can be risk-averse or risk-seeking. The latter is much scarcer in the literature and can be taken as a starting point for future research efforts. It also presents more difficult situations under this model given the complexity of the conditions that have to hold for the performance measure to behave appropriately. However, we show that when those conditions apply, the impact of a risk-seeking newsvendor on the objective function can be remarkable.

We present results that back those found for risk-averse situations in other works with different models, plus we add conclusions for risk-seeking cases along with other findings that, despite of being intuitive, need mathematical endorsement. For instance, it was shown that the optimal price for a given safety stock z, $p^*(z)$, is smaller in risk-averse cases than in risk-neutral cases. Conversely, in risk-seeking cases, this price is greater. In both scenarios, however, it is an increasing function of z. This price is also concave in z for the risk-averse case, whereas such concavity is guaranteed in a smaller range of λ for the risk-seeking case.

Furthermore, it was found that the optimal safety stock as a function of the price, $z^*(\cdot)$, always decreases as we face an increase in risk-aversion. Intuitively, one might think that the opposite would happen in risk-seeking cases (i.e., $\tilde{z}^*(\cdot)$ always increases as we turn to be more risk-seeking). However, this result is true only provided that $\hat{\varepsilon}^*(p) \geq bp\frac{1}{B - E(\epsilon)}$.

Finally, we comment on the values of λ. The restrictions that our models set on the range of values that this parameter may take have to be considered as a scaling measure of risk. Out of the proposed ranges for λ, the concavity of the objective function is not guaranteed, although it might occur. In particular, if we are concerned about a risk-averse environment, it makes sense to set the most averse case to the maximum value that λ can take (i.e., $\frac{1}{4p_{max}(B - E(\epsilon))}$) and scale our risk situations according to the range $\left[0, \frac{1}{4p_{max}(B - E(\epsilon))}\right)$. Similarly, we can identify the most favorable risk-seeking case according to the lower bounds described on *Theorem 3* or *Theorem 4*, whatever suits us best, and evaluate different risk measures according to this scale. Having such bounds on the risk parameter is reasonable. It is seldom that the results hold for the entire range of λ . Even in models with the exponential utility function (in which the decision maker is equipped with a constant risk coefficient) or other measures of risk, the degree of risk aversion is somewhat bounded to obtain certain results. On the other hand, in real life problems, λ is usually small, or close to zero. Hence the results for large λ are not meaningful.

The model proposed here also has some limitations which in turn provide research opportunities for the future. In particular, the demand is linear with respect to the price. Other relationships between price and demand that will yield different conditions for the concavity of the objective function can be proposed. It would also be interesting to see how the results found in this paper change when the risk measure is different. The standard semideviation proposed in [21] is a reasonable alternative, although the analysis of the optimality conditions for the concavity of the objective function seems more complicated in this case. Further research can also be done if we consider censored demand that originates when there is a loss of information that does not allow us to observe the true demand. This occurs, for example, in the case of stockouts. Recent research efforts have been made in this area (see [22, 23, 24, 25]) from which new approaches and ideas can be drawn from. For more comments on research and an excellent survey the reader is referred to [26].

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Appendix A. Tables for the numerical examples in risk-averse scenarios

	$\lambda = 0$					$\lambda = 1/11200$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	21.49	0.60	106.04	106.04	70.23	21.45	0.50	105.60	106.03	69.34
Uniform $[-10, 10]$	21.04	0.66	101.77	101.77	74.51	21.21	0.54	101.28	101.76	73.34
Uniform $[-3, 40]$	21.25	19.76	129.46	129.46	157.73	21.13	19.34	127.95	129.42	153.45
	$\lambda = 1/5600$					$\lambda = 1/2800$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	21.41	0.41	105.18	106.02	68.46	21.33	0.23	104.36	105.96	66.78
Uniform $[-10, 10]$	21.31	0.42	100.81	101.74	72.19	21.36	0.19	99.91	101.66	70.00
Uniform $[-3, 40]$	21.02	18.93	126.52	129.30	149.39	20.83	18.17	123.88	128.90	141.86
	$\lambda = 1/1400$									
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$					
Trunc. $N(0, 100)$	21.19	-0.11	102.85	105.74	63.62					
Uniform $[-10, 10]$	21.41	-0.24	98.26	101.37	65.97					
Uniform $[-3, 40]$	20.49	16.84	119.32	127.61	128.93					

Table A1: Optimum values of $P(p, z)$ as a function of λ.

	$\lambda = 0$				$\lambda = 1/11200$				$\lambda = 1/5600$			
Distribution	z^*	P^*	$E[P^*]$	$SD[P^*]$	z^*	P^*	$E[P^*]$	$SD[P^*]$	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(0, 100)$	0.00	104.01	104.01	60.89	-0.07	103.69	104.01	60.36	-0.14	103.36	104.00	59.83
Uniform $[-10, 10]$	0.00	100.00	100.00	64.55	-0.09	99.63	100.00	63.87	-0.18	99.27	99.98	63.15
Uniform $[-3, 40]$	18.5	127.50	127.50	138.78	18.22	126.32	127.48	136.61	17.94	125.18	127.43	134.43
	$\lambda = 1/2800$				$\lambda = 1/1400$							
Distribution	z^*	P^*	$E[P^*]$	$SD[P^*]$	z^*	P^*	$E[P^*]$	$SD[P^*]$				
Trunc. $N(0, 100)$	-0.27	102.73	103.97	58.84	-0.53	101.54	103.85	56.87				
Uniform $[-10, 10]$	-0.34	98.57	99.94	61.91	-0.66	97.26	99.78	59.40				
Uniform $[-3, 40]$	17.41	122.99	127.22	130.30	16.44	119.01	126.51	122.70				

Table A2: Behavior of $z^*(20)$ as a function of λ.

Appendix B. Tables for the numerical examples in risk-seeking scenarios

	$\lambda = 0$					$\lambda = \lambda_{min}/4.5$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(25, 1600)$	3.93	10.75	37.80	37.80	40.45	3.99	15.84	40.13	37.32	48.66
Uniform $[-30, 200]$	8.57	12.11	120.68	120.68	82.79	8.60	16.36	122.57	120.39	95.23
	$\lambda = \lambda_{min}/3$					$\lambda = \lambda_{min}/1.5$				
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$
Trunc. $N(25, 1600)$	4.04	19.23	41.70	36.44	54.37	4.27	34.38	49.70	26.01	81.43
Uniform $[-30, 200]$	8.62	19.20	123.75	119.88	103.80	8.75	35.01	129.38	112.13	154.97
	$\lambda = \lambda_{min}$									
Distribution	p^*	z^*	P^*	$E[P^*]$	$SD[P^*]$					
Trunc. $N(25, 1600)$	4.63	52.40	67.21	-3.48	115.05					
Uniform $[-30, 200]$	9.37	103.74	156.05	-28.85	414.25					

Table B1: Optimum values of $P(p, z)$ as a function of λ.

Appendix C. Figures for the numerical examples in risk-averse scenarios

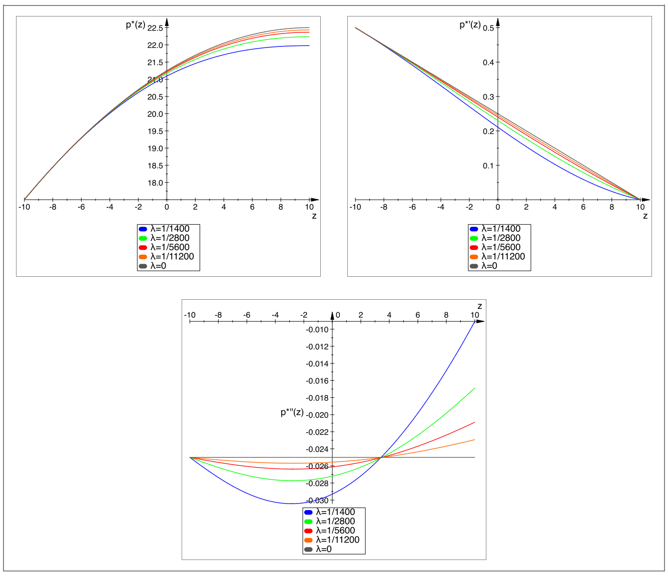

Figure C1: $p^*(z)$ and its first two derivatives for a uniform distribution in $[-10, 10]$

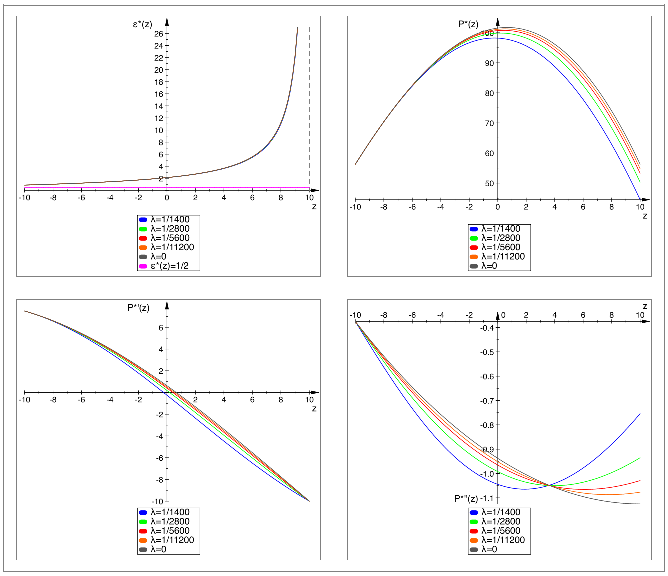

Figure C2: $\tilde{\varepsilon}^*(z)$, $P^*(z)$ and its first two derivatives for a uniform distribution in $[-10, 10]$

Appendix D. Figures for the numerical examples in risk-seeking scenarios

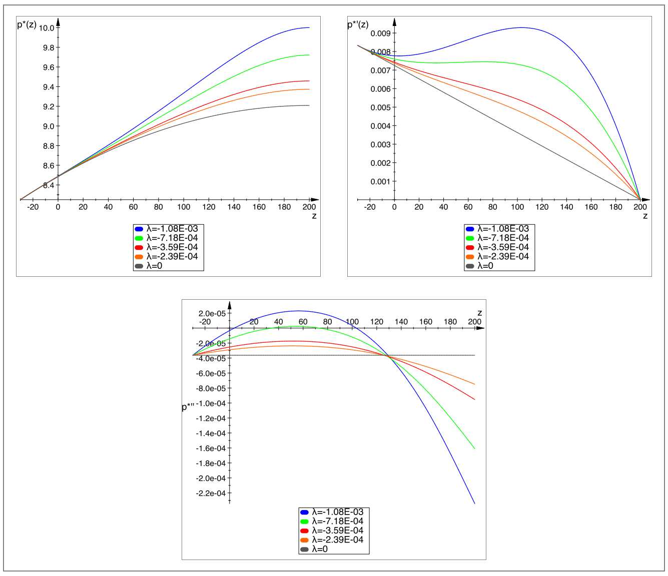

Figure D1: $p^*(z)$ and its first two derivatives for a uniform distribution in $[-30, 200]$

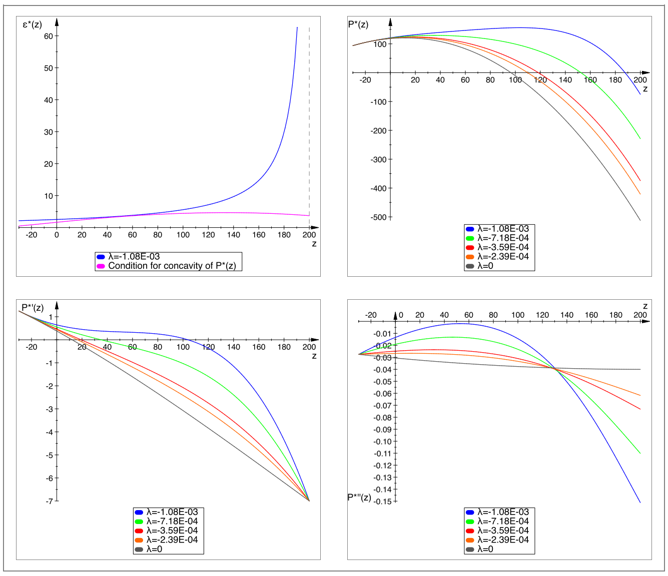

Figure D2: Test of $\tilde{\varepsilon}^*(z)$, $P^*(z)$ and its first two derivatives for a uniform distribution in $[-30, 200]$

Appendix E. Proofs of selected theorems and lemmas

Theorem 1

Proof. First, we show that $p^*(\cdot)$ is concave. Indeed, the second-order derivative of this function yields

$$\begin{aligned}\frac{d^2p^*(z)}{dz^2} &= \frac{-f(z)(\lambda\sigma^2(z)+b)-2\lambda(z-\mu(z))(1-F(z))^2}{2(\lambda\sigma^2(z)+b)^2} \\ &\quad -4\lambda\frac{1-F(z)}{2(\lambda\sigma^2(z)+b)}\Big[F(z)*p^*(z)+(z-\mu(z))\frac{dp^*(z)}{dz}\Big], \qquad \text{(E.1)}\end{aligned}$$

which is clearly nonpositive, since $z-\mu(z)\geq 0$ for $z\in[A,B]$ and the function $p^*(\cdot)$ is increasing.

Analyzing (8) at the extreme points of the interval $[A,B]$ yields:

$$\left.\frac{dP^*(z)}{dz}\right|_{z=A} = \frac{A+a+cb}{2b}-\frac{2bc}{2b} > 0, \quad \text{by assumption (A5),} \qquad \text{(E.2)}$$

$$\left.\frac{dP^*(z)}{dz}\right|_{z=B} = -c < 0. \qquad \text{(E.3)}$$

Therefore, there exists a point $z^*\in(A,B)$ at which the function $P^*(\cdot)$ attains its maximum. We claim that such a point is unique by showing that $P^*(\cdot)$ is a concave function. Indeed,

$$\begin{aligned}\frac{d^2P^*(z)}{dz^2} &= \left(\frac{dp^*(z)}{dz}(1-F(z))-f(z)p^*(z)\right)[1-2\lambda(z-\mu(z))p^*(z)] \\ &\quad -2\lambda p^*(z)(1-F(z))\left[F(z)p^*(z)+(z-\mu(z))\frac{dp^*(z)}{dz}\right]. \qquad \text{(E.4)}\end{aligned}$$

Note the similarity of the last term on the right-hand side to the right hand side of eq.(E.1); hence this term is also nonpositive. The first term of the right-hand side,however, might take on a different sign. While (A3) guarantees that

$[1 - 2\lambda(z - \mu(z))p^*(z)] \geq 0$, it is unclear what happens with the first part of the term. If we force it to be nonpositive, we have that

$$\frac{dp^*(z)}{dz}(1 - F(z)) - f(z)p^*(z) \quad \leq \quad 0.$$

However, since the function $z \mapsto \frac{dp^*(z)}{dz}$ is decreasing, it attains its maximum at $z = A$:

$$\left.\frac{dp^*(z)}{dz}\right|_{z=A} = \frac{1}{2b} \implies \frac{dp^*(z)}{dz}(1 - F(z)) - f(z)p^*(z) \quad \leq \frac{1}{2b}(1 - F(z)) - f(z)p^*(z) \leq 0.$$

The last inequality is equivalent to

$$\frac{bp^*(z)f(z)}{1 - F(z)} \quad \geq \quad \frac{1}{2},$$

from which we conclude that $P^*(\cdot)$ is concave if $\tilde{\varepsilon}^*(z) \geq \frac{1}{2}, \forall z \in [A, B]$. □

Lemma 2

Proof. Given the complexity of eq.(12), we proceed to see how $\tilde{z}^*$ varies with changing λ. Thus, if we rename the left-hand side of (12) as $g(\lambda, z)$, the following holds

$$\begin{aligned} \frac{\partial g(\lambda, z)}{\partial \lambda} &= -2p(1 - F(z)[z - \mu(z)], \\ \frac{\partial g(\lambda, z)}{\partial z} &= f(z)[2\lambda p(z - \mu(z)) - 1] - 2\lambda pF(z)(1 - F(z)). \end{aligned}$$

By means of the *Implicit Function Theorem* [27] it turns out that

$$\frac{d\tilde{z}^*(\lambda)}{d\lambda} = \frac{2p(1 - F(\tilde{z}^*(\lambda))[\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))]}{f(\tilde{z}^*(\lambda))[2\lambda p(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))) - 1] - 2\lambda pF(\tilde{z}^*(\lambda))(1 - F(\tilde{z}^*(\lambda)))}. \tag{E.5}$$

The numerator in the formula above is always nonnegative. The second term of the denominator is always nonnegative as well but it is subtracted. Then, if the first term in the denominator is negative, the entire expression will become negative. This occurs, if:

$$2\lambda p(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))) - 1 \quad < \quad 0,$$

whence

$$\lambda < \quad \frac{1}{2p(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda)))} \quad \leq \frac{1}{2p_{max}(B - E(\epsilon))},$$

but this is guaranteed by (A3). Therefore, $\tilde{z}^*(\cdot)$ is decreasing. □

Theorem 2

Proof. We must show that the Hessian matrix of $P(\cdot)$ is negative semidefinite. This implies that

$$\frac{\partial^2 P(p,z)}{\partial z^2} \leq 0 \quad \text{and} \quad \Delta(p,z) = \frac{\partial^2 P(p,z)}{\partial p^2}\frac{\partial^2 P(p,z)}{\partial z^2} - (\frac{\partial^2 P(p,z)}{\partial p \partial z})^2 \geq 0.$$

Note that the validity of the conditions above also implies that $\frac{\partial^2 P(p,z)}{\partial p^2} \leq 0$. Equations (4) and (11) are negative and nonnegative respectively. Likewise, the second-order partial derivative of $z \mapsto P(p,z)$ with respect to z is nonpositive as a consequence of a straightforward application of (A3):

$$\frac{\partial^2 P(p,z)}{\partial z^2} = -2\lambda p^2 F(z)[1-F(z)] - pf(z)[1-2\lambda p(z-\mu(z))] \leq 0.$$

Finally, it remains to check that the determinant of the Hessian matrix is nonnegative:

$$\begin{aligned}
\Delta(p,z) &= 2[\lambda\sigma^2(z)+b][2\lambda p^2 F(z)(1-F(z)) + pf(z)(1-2\lambda p(z-\mu(z)))] \\
&\quad -[1-F(z)]^2[1-4\lambda p(z-\mu(z))]^2]] \\
&\geq (1-F(z))\Big\{\Big(4\lambda b p^2 F(z) + (1-2\lambda p(z-\mu(z)))\Big) \\
&\quad +F(z)[1-4\lambda p(z-\mu(z))]^2 - [1-4\lambda p(z-\mu(z))]^2\Big\} \geq 0,
\end{aligned}$$

where the inequality follows from the fact that $2(\lambda\sigma^2(z)+b) \geq 2b$ and from assuming that $\tilde{\varepsilon}(p,z) \geq \frac{1}{2}$. □

Lemma 4

Proof. Given (5), it only remains to prove that $c < p^*(z) \leq p_{max}$. Indeed, if $\lambda \in \left(\frac{-b}{Var(\epsilon)}, 0\right)$, the right-hand side of (6) is always positive and thus $p^*(\cdot)$ is increasing. Besides, $p^*(A) = \frac{A+a+cb}{2b} > c$. Therefore, $p^*(z) > c,\ \forall z \in [A,B]$.

Nonetheless, as shown before, $p^*(z)$ may take values greater than p_{max}. Hence, if we require $p^*(B) \leq p_{max} \leq \frac{a}{b}$, then $p^*(z) \leq p_{max}$ for any $z \in [A,B]$ (because $p^*(\cdot)$ is increasing). Consequently,

$$p^*(B) = \frac{E(\epsilon)+a+cb}{2(\lambda Var(\epsilon)+b)} \leq \frac{a}{b},$$

which holds whenever

$$\lambda \geq \frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)},$$

with the right-hand side being negative, as guaranteed by (B4). Now, it remains to require the number $\frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)}$ being contained in the interval $\left(\frac{-b}{Var(\epsilon)}, 0\right)$. This occurs, whenever

$$\frac{b(E(\epsilon)-y(c))}{2aVar(\epsilon)} \geq \frac{-b}{Var(\epsilon)},$$

whence we obtain the necessary condition $E(\epsilon) \geq -y(-c)$. This condition, however, is always met by virtue of assumption (B3) and the fact that $E(\epsilon) > A$:

$$-y(c) < A < E(\epsilon) \quad \Longrightarrow \quad -y(-c) < E(\epsilon) - 2bc < E(\epsilon).$$

Therefore, for $\lambda \in \left[\frac{b(E(\epsilon) - y(c)}{2aVar(\epsilon)}, 0\right)$ the function $P(\cdot, z)$ is concave for all $z \in [A, B]$ and $p^*(z) \leq p_{max}$. This concludes the proof. □

Theorem 3

Proof. The first-order derivative of $P^*(\cdot)$ at the points A and B is given by (E.2) and (E.3), respectively. Therefore, there exists a point $z^* \in (A, B)$ at which the function $P^*(\cdot)$ attains its maximum. Such maximum is unique if $P^*(\cdot)$ is a concave function and we refer to equation (E.4) in order to prove it. Imposing concavity requires that

$$\begin{aligned}[1 - 2\lambda(z - \mu(z))p^*(z)]\left(\tfrac{dp^*(z)}{dz}[1 - F(z)] - f(z)p^*(z)\right) \quad &\leq \\ 2\lambda p^*(z)[1 - F(z)]\left(F(z)p^*(z) + (z - \mu(z))\tfrac{dp^*(z)}{dz}\right).&\end{aligned}$$

This condition holds for $z = B$. For all other values of z, dividing both sides by $1 - F(z)$ and multiplying by b gives an expression in terms of $\tilde{\varepsilon}^*(z)$:

$$\tilde{\varepsilon}^*(z) \quad \geq \quad b\frac{dp^*(z)}{dz} - \frac{2\lambda p^*(z)b\left[F(z)p^*(z) + (z - \mu(z))\frac{dp^*(z)}{dz}\right]}{1 - 2\lambda(z - \mu(z))p^*(z)}, \quad z \in [A, B],$$

which is a necessary and sufficient condition for $P^*(\cdot)$ to be concave. □

Lemma 5

Proof. A straightforward application of (E.1) gives a necessary condition for $\frac{dp^*(\cdot)}{dz}$ to be decreasing:

$$\begin{aligned}\frac{-f(z)(\lambda\sigma^2(z) + b) - 2\lambda(z - \mu(z))(1 - F(z))^2}{\lambda\sigma^2(z) + b}[1 - 4\lambda(z - \mu(z))p^*(z)] \quad &\leq \\ 4\lambda(1 - F(z))\left[F(z)p^*(z) + (z - \mu(z))\frac{dp^*(z)}{dz}\right], \quad z \in [A, B].&\end{aligned}$$

Bounding both sides of this equation yields a sufficient condition for the concavity of $p^*(\cdot)$. The condition above always holds for $z = A$ and $z = B$ (the expression above becomes $-f(A) \leq 0$ and $-f(B)[1 - 4\lambda(B - E(\epsilon))p^*(B)] \leq 0$ respectively). For $A < z < B$, we establish that the largest value of the left-hand side has to be at most equal to the smallest value of the right-hand side. The largest value of the left-hand side is represented by the value that is closest to 0, which is $\frac{1}{b}(-2\lambda(B - E(\epsilon)) - f(z)(\lambda Var(\epsilon) + b))$. Conversely, the smallest value of the right-hand side is $2\lambda\frac{2a + B - E(\epsilon)}{b}$. Therefore, we obtain that

$$\frac{1}{b}[-2\lambda(B - E(\epsilon)) - f(z)(\lambda Var(\epsilon) + b)] \quad \leq \quad 2\lambda\frac{2a + B - E(\epsilon)}{b},$$

whence

$$\lambda \quad \geq \quad \frac{-f(z)b}{4(a+B-E(\epsilon))+f(z)Var(\epsilon)}.$$

Since we want this range to be valid for all z, we can rewrite this as

$$\lambda \quad \geq \quad \max_z \left\{ \frac{-f(z)b}{4(a+B-E(\epsilon))+f(z)Var(\epsilon)} \right\}, \quad z \in [A, B].$$

□

Lemma 6

Proof. We analyze again (E.5) under the light of the implicit function theorem, but this time with the condition $\lambda < 0$. The numerator in this equation is still nonnegative, but the second term in the denominator is now nonpositive and is being subtracted. The first term of the denominator is always nonpositive. Hence, if $\tilde{z}^*(\lambda)$ is to decrease with λ we must have that

$$f(\tilde{z}^*(\lambda))[2\lambda p(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))) - 1] - 2\lambda p F(\tilde{z}^*(\lambda))(1 - F(\tilde{z}^*(\lambda))) \quad < \quad 0,$$

but we know that this surely occurs in $p = p_A$ and $p = p_B$, for if there exists $p = p_A$ such that $\tilde{z}^*(\lambda) = A$ this condition simplifies to $-f(A) \leq 0$, which always holds. Moreover, if there exists $p = p_B$ such that $\tilde{z}^*(\lambda) = B$ this condition requires $2\lambda p(B - E(\epsilon)) - 1 \leq 0$, or equivalently $\lambda \leq \frac{1}{2p(B-E(\epsilon))}$, which also holds. For all other values of p we can assert that $\frac{d\tilde{z}^*(\lambda)}{d\lambda}$ is nonpositive if

$$\lambda \quad \leq \quad \frac{f(\tilde{z}^*(\lambda))}{2p\left[f(\tilde{z}^*(\lambda))(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))) - F(\tilde{z}^*(\lambda))(1 - F(\tilde{z}^*(\lambda)))\right]},$$

where $c < p \leq p_{max}$ and $p \neq p_A, p_B$. If the right-hand side of this equation is positive, then the condition above always holds and $\frac{d\tilde{z}^*(\cdot)}{d\lambda} \leq 0$. This right-hand side is positive as long as

$$f(\tilde{z}^*(\lambda))(\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))) - F(\tilde{z}^*(\lambda))(1 - F(\tilde{z}^*(\lambda))) \quad > \quad 0,$$

whence, after dividing by $1 - F(\tilde{z}^*(\lambda))$ and multiplying by bp we obtain that

$$\hat{\varepsilon}^*(p) \quad > \quad bp\frac{F(\tilde{z}^*(\lambda))}{\tilde{z}^*(\lambda) - \mu(\tilde{z}^*(\lambda))}, \quad c < p \leq p_{max}, \; p \neq p_A, p_B.$$

An upper bound of the right-hand side is given by $bp\frac{1}{B-E(\epsilon)}$Therefore, we can conclude that if

$$\hat{\varepsilon}^*(p) > bp\frac{1}{B-E(\epsilon)}, \quad c < p \leq p_{max},$$

then, given a price p, $\tilde{z}^*(\cdot)$ decreases. In other words, when λ decreases, i.e., as we focus on more risk-seeking situations, $\tilde{z}^*(\cdot)$ increases. This concludes the proof. □